Fatou's Theorem and minimal graphs

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Abstract

In this paper we extend a recent result of Collin-Rosenberg (*a solution for the minimal surface equation in the Euclidean disc has radial limits almost everywhere*) for a large class of differential operators in Divergence form. Also, we give an alternative proof of Fatou's Theorem (*a harmonic function defined in the Euclidean disc has radial limits almost everywhere*) even for harmonic functions that are not bounded. Moreover, we construct an example (in the spirit of [3]) of a minimal graph in $\mathbb{M}^2 \times \mathbb{R}$, where $\mathbb{M}^2$ is a Hadamard surface, over a geodesic disc which has finite radial limits in a mesure zero set.

1 Introduction

It is well known that a bounded harmonic function u defined on the Euclidean disc D has radial limits almost everywhere (Fatou's Theorem [4]). Moreover, the radial limits can not be plus infinity for a positive measure set. For fixed $\theta \in \mathbb{S}^{n-1}$, the radial limit $u(\theta)$ (if it exists) is defined as

$$u(\theta) = \lim_{r \to 1} u(r, \theta),$$

where we paramatrize the Euclidean disc in polar coordinates $(r, \theta) \in [0, 1) \times \mathbb{S}^1$.

In 1965, J. Nitsche [8] asked if a Fatou Theorem is valid for the minimal surface equation, i.e., *does a solution for the minimal surface equation in the Euclidean disc have radial limits almost everywhere?* This question has been solved recently by P. Collin and H. Rosenberg

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[3]. Moreover, in the same paper [8], J. Nitsche asked: *what is the largest set of θ for which a minimal graph on D may not have radial limits?* Again, this question was solved in [3] if one allows infinite radial limits. That is, they construct an example of a minimal graph in the Euclidean disc with finite radial limits only on a set of measure zero. In this example, the $+\infty$ radial limits (resp. $-\infty$) are taken on a set of measure π (resp. π).

The aim of this paper is to extend both results and give an alternative proof of Fatou's Theorem for a more general situation. In Section 2, we extend Collin-Rosenberg's Theorem for a large class of differential operators in divergence form (see Theorem 2.1). Also, we extend Fatou's Theorem even for harmonic functions that are not bounded (see Theorem 2.2). In particular, as a consequence of this result, we obtain the classical Fatou Theorem (see Corollary 2.1). In Section 3, we construct an example of a minimal graph in $\mathbb{M}^2 \times \mathbb{R}$ over a geodesic disk $\mathcal{D} \subset \mathbb{M}^2$ ($\mathbb{M}^2$ is a Hadamard surface) for which the finite radial limits are of measure zero. Also, the $+\infty$ radial limits (resp. $-\infty$) are taken on a set of measure π (resp. π).

2 Fatou's Theorem

Henceforth $(\mathbb{B}, g)$ denotes the $n-$dimensional unit open ball, i.e,

$$\mathbb{B} = \left\{(r, \theta)\,;\, 0 \le r < 1,\, \theta \in \mathbb{S}^{n-1}\right\},$$

in polar coordinates with respect to g, g a C^2-Riemannian metric on $\mathbb{B}$. Define $G := G(r,\theta) = \sqrt{\det(g)}$. Moreover, we denote by ∇ the Levi-Civita connection associated to g and by div_g its associated divergence operator. Also, $L^1(\mathbb{B})$ denotes the set of integrable functions on $(\mathbb{B}, g)$.

Set $u \in C^2(\mathbb{B})-$function and X_u be a $C^1(\mathbb{B})-$vector field so that its coordinates depend on u, its first derivatives and $C^1(\mathbb{B})-$functions.

For fixed $\theta \in \mathbb{S}^{n-1}$, the radial limit (if it exists) $u(\theta)$ is defined as

$$u(\theta) = \lim_{r \to 1} u(r, \theta).$$

Theorem 2.1. *Let $(\mathbb{B}, g, G, u, X_u)$ be as above. Assume that*

a) $\alpha \le G(r,\theta) \le \beta$ for all $(r,\theta) \in [0,1) \times \mathbb{S}^{n-1}$, α and β positive constants.

b) $|X_u| \le M$ on $\mathbb{B}$, i.e., X_u is bounded on $\mathbb{B}$.

c) $g(\nabla u, X_u) \ge \delta\,|\nabla u| + h$, where δ is a positive constant and $h \in L^1(\mathbb{B})$.

Let $f \in L^1(\mathbb{B})$. If u is a solution of

$$\mathrm{div}_g(X_u) \ge \ (\ \textit{or}\ \ \le)\ f\ \ \textit{on}\ \mathbb{B},$$

then u has radial limits almost everywhere.

Proof. First, let us prove the case

$$\mathrm{div}_g(X_u) \geq \ f.$$

For $r < 1$ fixed, set $\mathbb{B}(r)$ the $n-$dimensional open ball of radius r. Let $\eta : \mathbb{R} \longrightarrow (0, +1)$ be a smooth function so that $0 < \eta'(x) < 1$ for all $x \in \mathbb{R}$. Define $\psi := \eta \circ u$.

On the one hand, by direct computations and *item c)*, we have

$$\begin{aligned}
\mathrm{div}_g(\psi\, X_u) &= \psi\, \mathrm{div}_g(X_u) + g(\nabla\psi, X_u) \geq \psi\, f + \eta'\, g(\nabla u, X_u) \\
&\geq \psi\, f + \eta'\, (\delta\, |\nabla u| + h) = \delta\, \eta'\, |\nabla u| + (\psi\, f + \eta'\, h) \\
&= \delta\, |\nabla \psi| + (\psi\, f + \eta'\, h)\,,
\end{aligned}$$

thus

$$\int_{\mathbb{B}(r)} \mathrm{div}_g(\psi\, X_u) \geq \delta \int_{\mathbb{B}(r)} |\nabla\psi| + C \tag{2.1}$$

where C is some constant. This follows since f and h are L^1-functions on $\mathbb{B}$.

On the other hand, by Stokes' Theorem and *items a)* and b), we obtain for $r < 1$ fixed

$$\begin{aligned}
\int_{\mathbb{B}(r)} \mathrm{div}_g(\psi\, X_u) &= \int_{\partial\mathbb{B}(r)} \psi\, g(X_u, \upsilon) \leq \int_{\partial\mathbb{B}(r)} M \\
&= M \int_{\theta\in\mathbb{S}^{n-1}} G(r,\theta) d\theta \leq M\, \beta \int_{\theta\in\mathbb{S}^{n-1}} \\
&= M\, \beta\, \omega_{n-1},
\end{aligned} \tag{2.2}$$

where υ is the outer conormal to $\partial\mathbb{B}(r)$ and ω_{n-1} is the volume of $\mathbb{S}^{n-1}$.

So, from (2.1), (2.2) and letting r go to one, we conclude that $|\nabla\psi|$ is integrable in $\mathbb{B}$, i.e.,

$$\int_{\mathbb{B}} |\nabla\psi| < +\infty \tag{2.3}$$

Since $\dfrac{\partial\psi}{\partial r} \leq |\nabla\psi|$, we have from Fubini's Theorem and (2.3)

$$\int_{\mathbb{B}} \frac{\partial\psi}{\partial r} = \int_{\theta\in\mathbb{S}^{n-1}} \left(\int_0^1 \frac{\partial\psi}{\partial r} G(r,\theta)\, dr \right) d\theta < \infty.$$

Thus, as $G(r,\theta)$ is bounded below by a positive constant, for almost all $\theta \in \mathbb{S}^{n-1}$,

$$\lim_{r\to 1} \psi(r,\theta) - \psi(0,0) = \int_0^1 \frac{\partial\psi}{\partial r}(r,\theta) dr < \infty,$$

that is, ψ has radial limits almost everywhere. Since $\psi = \eta \circ u$, we conclude u has radial limits almost everywhere (which may be $\pm\infty$).

For

$$\operatorname{div}_g(X_u) \leq f,$$

we just have to follow the above proof by changing $\eta : \mathbb{R} \longrightarrow (-1, 0)$ so that $0 < \eta'(x) < 1$ for all $x \in \mathbb{R}$. ☐

As we pointed out in the Introduction, in the spirit of Theorem 2.1, we can give an alternative proof of Fatou's Theorem even for harmonic function that are not bounded, i.e.,

Theorem 2.2. *Let $(\mathbb{B}, g, G, u)$ be as above. Assume that $\alpha \leq G(r, \theta) \leq \beta$ for all $(r, \theta) \in [0, 1) \times \mathbb{S}^{n-1}$, α and β positive constants. If u is a solution of*

$$\operatorname{div}_g(\nabla u) = 0 \text{ on } \mathbb{B},$$

then u has radial limits almost everywhere.

Proof. For $r < 1$ fixed, set $\mathbb{B}(r)$ the $n-$dimensional open ball of radius r. Let $\eta : \mathbb{R} \longrightarrow (0, 1)$ be a smooth function so that $0 < \eta'(x) < 1$ for all $x \in \mathbb{R}$. Define

$$\phi := \eta \circ u, \text{ and } \psi := \frac{\phi}{\sqrt{1 + |\nabla u|^2}}.$$

On the one hand, by direct computations, we have

$$\begin{aligned}
\operatorname{div}_g(\psi \nabla u) &= \psi \operatorname{div}_g(\nabla u) + \langle \nabla \psi, \nabla u \rangle = \langle \nabla \psi, \nabla u \rangle \\
&= \eta' \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} - \phi \frac{\langle \nabla |\nabla u|^2, \nabla u \rangle}{2(1 + |\nabla u|^2)^{3/2}} \\
&\geq \eta' |\nabla u| - \frac{\eta'}{\sqrt{1 + |\nabla u|^2}} - \phi \frac{\langle \nabla |\nabla u|^2, \nabla u \rangle}{2(1 + |\nabla u|^2)^{3/2}} \\
&\geq |\nabla \phi| - 1 - \frac{\langle \nabla |\nabla u|^2, \nabla u \rangle}{2(1 + |\nabla u|^2)^{3/2}},
\end{aligned}$$

since

$$\nabla \psi = \eta' \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} - \phi \frac{\nabla |\nabla u|^2}{2(1 + |\nabla u|^2)^{3/2}}.$$

Let us first bound the term

$$\left| \int_{\mathbb{B}(r)} \frac{\langle \nabla |\nabla u|^2, \nabla u \rangle}{2(1 + |\nabla u|^2)^{3/2}} \right|.$$

Set $Y_u := \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}$, then

$$\begin{aligned}\operatorname{div}_g(Y_u) &= \frac{1}{\sqrt{1+|\nabla u|^2}}\operatorname{div}_g(\nabla u) - \frac{\langle \nabla |\nabla u|^2, \nabla u\rangle}{2(1+|\nabla u|^2)^{3/2}}\\ &= -\frac{\langle \nabla |\nabla u|^2, \nabla u\rangle}{2(1+|\nabla u|^2)^{3/2}}\end{aligned}$$

since $\operatorname{div}_g(\nabla u) = 0$. Applying Stoke's Theorem we obtain

$$\int_{\mathbb{B}(r)} \operatorname{div}_g(Y_u) = \int_{\partial\mathbb{B}(r)} \langle Y_u, \upsilon\rangle,$$

that is

$$\begin{aligned}\left|\int_{\mathbb{B}(r)} \frac{\langle \nabla |\nabla u|^2, \nabla u\rangle}{2(1+|\nabla u|^2)^{3/2}}\right| = \left|\int_{\partial\mathbb{B}(r)} \langle Y_u, \upsilon\rangle\right| &\leq \int_{\partial\mathbb{B}(r)} |Y_u|\\ &= \int_{\partial\mathbb{B}(r)} \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}} \leq \int_{\partial\mathbb{B}(r)} \leq C,\end{aligned}$$

for some positive constant C.

Thus

$$\int_{\mathbb{B}(r)} \operatorname{div}(\psi\, X_u) \geq \int_{\mathbb{B}(r)} |\nabla\phi| - \tilde{C}, \tag{2.4}$$

for some positive constant $\tilde{C}$.

On the other hand, by Stokes' Theorem we obtain for $r < 1$ fixed

$$\begin{aligned}\int_{\mathbb{B}(r)} \operatorname{div}(\psi\, X_u) = \int_{\partial\mathbb{B}(r)} \psi\,\langle X_u, \upsilon\rangle &\leq \int_{\partial\mathbb{B}(r)} \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}}\\ &\leq \int_{\partial\mathbb{B}(r)} \leq C',\end{aligned} \tag{2.5}$$

where υ is the outer conormal to $\partial\mathbb{B}(r)$ and C' is some positive constant.

So, from (2.4), (2.5) and letting r go to one, we conclude that $|\nabla\phi|$ is integrable in $\mathbb{B}$, i.e.,

$$\int_{\mathbb{B}} |\nabla\phi| < \infty \tag{2.6}$$

Since $\dfrac{\partial \phi}{\partial r} \leq |\nabla \phi|$, we have from Fubini's Theorem and (2.6)

$$\int_{\mathbb{B}} \frac{\partial \phi}{\partial r} = \int_{\theta \in \mathbb{S}^1} \left(\int_0^1 \frac{\partial \phi}{\partial r} G(r, \theta)\, dr \right) d\theta < \infty.$$

Thus, as $G(r, \theta)$ is bounded below by a positive constant, for almost all $\theta \in \mathbb{S}^{n-1}$,

$$\lim_{r \to 1} \phi(r, \theta) - \phi(0, 0) = \int_0^1 \frac{\partial \phi}{\partial r}(r, \theta) dr < \infty,$$

that is, ϕ has radial limits almost everywhere. Since $\phi = \eta \circ u$, we conclude u has radial limits almost everywhere (which may be $\pm\infty$). $\square$

Then, as a consequence

Corollary 2.1. *Let u be a harmonic function defined over the Euclidean disc. Then u has radial limits almost everywhere.*

2.1 Applications

Moreover, we will see now how Theorem 2.1 applies to get radial limits almost everywhere for minimal graphs in ambient spaces besides $\mathbb{R}^3$. We work here in Heisenberg space, but it is not hard to check that we could work with minimal graphs in a more general submersion (see [7]).

First, we need to recall some definitions in Heisenberg space (see [1]). The Heisenberg spaces are $\mathbb{R}^3$ endowed with a one parameter family of metrics indexed by bundle curvature by a real parameter $\tau \neq 0$. When we say the *Heisenberg space*, we mean $\tau = 1/2$, and we denote it by $\mathcal{H}$.

In global exponential coordinates, $\mathcal{H}$ is $\mathbb{R}^3$ endowed with the metric

$$g = (dx^2 + dy^2) + (\frac{1}{2}(ydx - xdy) + dz)^2.$$

The Heisenberg space is a Riemannian submersion $\pi : \mathcal{H} \longrightarrow \mathbb{R}$ over the standard flat Euclidean plane $\mathbb{R}^2$ whose fibers are the vertical lines, i.e., they are the trajectories of a unit Killing vector field and hence geodesics.

Let $S_0 \subset \mathcal{H}$ be the surface whose points satisfy $z = 0$. Let $D \subset \mathbb{R}^2$ be the unit disc. Henceforth, we identify domains in $\mathbb{R}^2$ with its lift to S_0. The Killing graph of a function $u \in C^2(D)$ is the surface

$$\Sigma = \{(x, y, u(x, y))\, ; \, (x, y) \in D\}.$$

Moreover, the minimal graph equation is

$$\mathrm{div}_{\mathbb{R}^2}(X_u) = 0,$$

here $\operatorname{div}_{\mathbb{R}^2}$ stands for the divergence operator in $\mathbb{R}^2$ with the Euclidean metric $\langle , \rangle$, and

$$X_u := \frac{\alpha}{W}\partial_x + \frac{\beta}{W}\partial_y,$$

where

$$\alpha := \frac{y}{2} + u_x, \quad \beta := -\frac{x}{2} + u_y,$$

and

$$W^2 = 1 + \alpha^2 + \beta^2.$$

Thus, for verifying u has radial limits almost everywhere (which may be $\pm\infty$), we have to check conditions $a)$, $b)$ and $c)$. Item $a)$ is immediate since we are working with the Euclidean metric.

Item $b)$ follows from

$$|X_u|^2 = \frac{\alpha^2 + \beta^2}{1 + \alpha^2 + \beta^2} \leq 1.$$

Now, we need to check Item $c)$. On one hand, using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned}
W^2 = 1 + \alpha^2 + \beta^2 &= 1 + u_x^2 + u_y^2 + (yu_x - xu_y) + \frac{x^2 + y^2}{4} \\
&= 1 + |\nabla u|^2 + \langle \nabla u, (-y, x)\rangle + \frac{x^2 + y^2}{4} \\
&\geq 1 + |\nabla u|^2 - |\nabla u||(-y, x)| + \frac{x^2 + y^2}{4} \\
&= 1 + |\nabla u|^2 - r|\nabla u| + \frac{r^2}{4}
\end{aligned}$$

thus,

$$W \geq \sqrt{1 + \left(|\nabla u| - \frac{r}{2}\right)^2} \geq \left| |\nabla u| - \frac{r}{2} \right|.$$

We need a lower bound for W in terms of $|\nabla u|$. To do so, we distinguish two cases:

Case $|\nabla u| \leq 5/4$: Since

$$1 - r\,|\nabla u| + \frac{r^2}{4} \geq 1 - \frac{5r}{4} + \frac{r^2}{4} \geq 0 \text{ for all } r \leq 1,$$

we obtain

$$W \geq \sqrt{|\nabla u|^2 + 1 - r\,|\nabla u| + \frac{r^2}{4}} \geq |\nabla u|\,.$$

Case $|\nabla u| > 5/4$: We already know that

$$W \geq \left| |\nabla u| - \frac{r}{2} \right|,$$

thus, for $|\nabla u| > 5/4$, it is easy to see that

$$\left| |\nabla u| - \frac{r}{2} \right| \geq \frac{3}{10} |\nabla u| \text{ for all } r \leq 1.$$

So, in any case, for $\delta = 3/10 > 0$

$$W \geq \delta |\nabla u| . \tag{2.7}$$

On the other hand,

$$\begin{aligned}
\langle \nabla u, X_u \rangle &= \frac{u_x^2 + u_y^2 + \frac{1}{2}(yu_x - xu_y)}{W} \\
&= \frac{1 + u_x^2 + u_y^2 + (yu_x - xu_y) + \frac{x^2+y^2}{4}}{W} - \frac{1 + \frac{1}{2}(yu_x - xu_y) + \frac{x^2+y^2}{4}}{W} \\
&= \frac{W^2}{W} + h = W + h \geq \delta |\nabla u| + h,
\end{aligned}$$

where we have used (2.7) and h denotes the $L^1(D)-$function

$$h = -\frac{1 + \frac{1}{2}(yu_x - xu_y) + \frac{x^2+y^2}{4}}{\sqrt{1 + u_x^2 + u_y^2 + (yu_x - xu_y) + \frac{x^2+y^2}{4}}},$$

that is, Item $c)$ is satisfied. So,

Corollary 2.2. *A solution for the minimal surface equation in the Heisenberg space defined over a disc has radial limits almost everywhere (which may be $\pm\infty$).*

3 An example in a Hadamard surface

The aim of this Section is to construct an example of a minimal graph in $\mathbb{M}^2 \times \mathbb{R}$ over a geodesic disk $\mathcal{D} \subset \mathbb{M}^2$ ($\mathbb{M}^2$ is a Hadamard surface) for which the finite radial limits are of measure zero.

We need to recall preliminary facts about graphs over a Hadamard surface (see [5] for details). Henceforth, $\mathbb{M}^2$ denotes a simply connected with Gauss curvature bounded above by a negative constant, i.e., $K_{\mathbb{M}^2} \leq c < 0$.

Let $p_0 \in \mathbb{M}^2$ and $\mathcal{D}$ be the the geodesic disk in $\mathbb{M}^2$ centered at p_0 of radius one. Re-scaling in the metric, we can assume that

$$\max\left\{K_{\mathbb{M}^2}(p)\,;\ p \in \overline{\mathcal{D}}\right\} = -1.$$

From the Hessian Comparison Theorem (see e.g. [6]), $\partial\mathcal{D}$ bounds a strictly convex domain. We assume that $\partial\mathcal{D}$ is smooth, otherwise we can work in a smaller disc. We identify $\partial\mathcal{D} = \mathbb{S}^1$ and orient it counter-clockwise.

We say that Γ is an *admissible polygon* in $\mathcal{D}$ if Γ is a Jordan curve in $\overline{\mathcal{D}}$ which is a geodesic polygon with an even number of sides and all the vertices in $\partial\mathcal{D}$. We denote by $A_1, B_1, \ldots, A_k, B_k$ the sides of Γ which are oriented counter-clockwise. Recall that any two sides can not intersect in $\mathcal{D}$. Set D the domain in $\mathcal{D}$ bounded by Γ. By $|A_i|$ (resp. $|B_j|$), we denote the length of such a geodesic arc.

Theorem 3.1 ([9]). *Let $\Gamma \subset \mathbb{M}^2$ be a compact polygon with an even number of geodesic sides $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$, in that order, and denote by D the domain with $\partial D = \Gamma$. The necessary and sufficient conditions for the existence of a minimal graph u on D, taking values $+\infty$ on each A_i, and $-\infty$ on each B_j, are the two following conditions:*

1. $\sum_{i=1}^n |A_i| = \sum_{i=1}^n |B_i|$,
2. *for each inscribed polygon P in D (the vertices of P are among the vertices of Γ) $P \neq D$, one has the two inequalities:*

$$2a(P) < |P| \text{ and } 2b(P) < |P|.$$

Here $a(P) = \sum_{A_j \in P} |A_j|$, $b(P) = \sum_{B_j \in P} |B_j|$ and $|P|$ is the perimeter of P.

The construction of this example follows the steps in [3, Section III], but here we have to be more careful in the choice of the first *inscribed square* and the *trapezoids*. We need to choose them as *symmetric* as possible.

Let us first explain how we take the *inscribed square*: Let $L = \mathrm{length}(\partial\mathcal{D})$ and $\gamma(x_0, x_1)$ be the geodesic arc in $\mathcal{D}$ joining $x_0, x_1 \in \partial\mathcal{D}$. Fix $x_0 \in \partial\mathcal{D}$ and let $\alpha : \mathbb{R}/[0, L) \longrightarrow \partial\mathcal{D}$ an arc-length parametrization of $\partial\mathcal{D}$ (oriented count-clockwise). Set $x_1 = \alpha(L/2)$. Consider $x_0^{\pm}(s) = \alpha(\pm s)$ and $x_1^{\pm}(s) = \alpha(L/2 \pm s)$ for $0 \leq s \leq L/2$ (c.f. Figure 1), and denote

$$\begin{aligned}
B_1(s) &= \gamma(x_0^+(s), x_1^-(s)) \\
A_1(s) &= \gamma(x_1^-(s), x_1^+(s)) \\
B_2(s) &= \gamma(x_1^+(s), x_0^-(s)) \\
A_2(s) &= \gamma(x_0^-(s), x_1^+(s)).
\end{aligned}$$

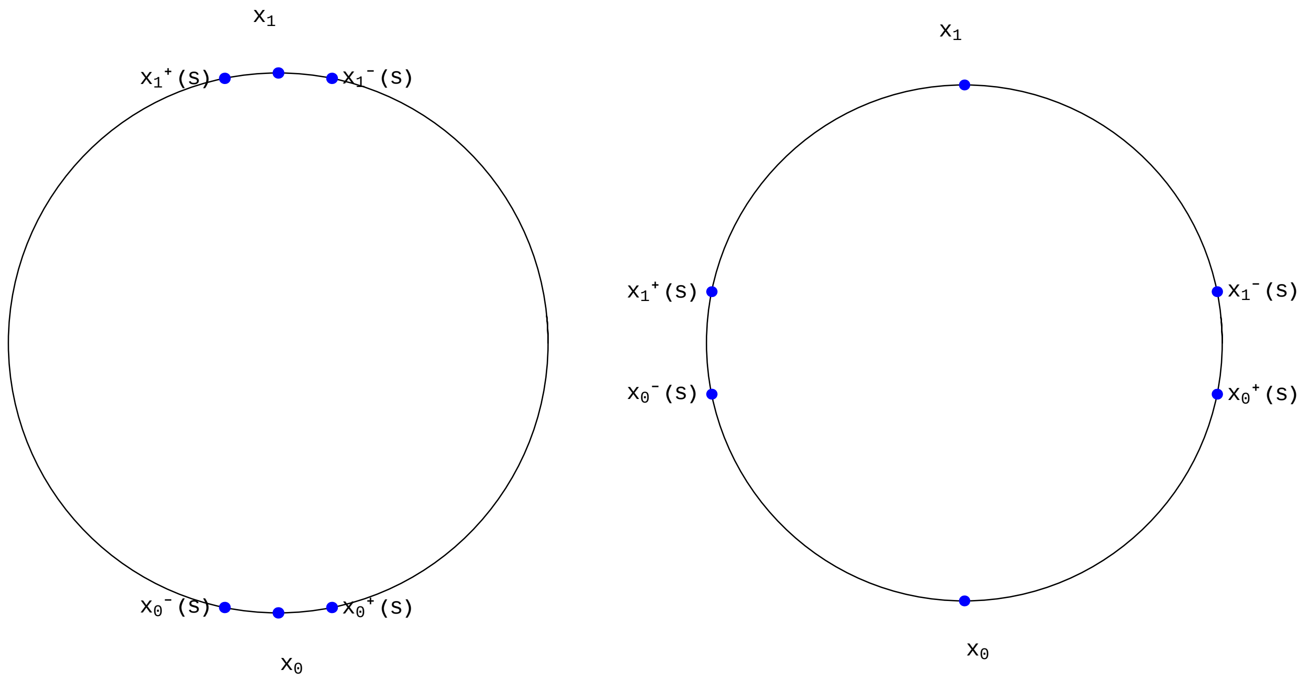


Figure 1: We move the points along $\partial\mathcal{D}$

Hence (c.f. Figure 2),

$$\begin{array}{rcll} |A_1(s)| + |A_2(s)| & > & |B_1(s)| + |B_2(s)| & \text{for } s \text{ close to } 0. \\ |A_1(s)| + |A_2(s)| & < & |B_1(s)| + |B_2(s)| & \text{for } s \text{ close to } L/2. \end{array}$$

Thus, there exist $s_0 \in (0, L/2)$ so that

$$|A_1(s_0)| + |A_2(s_0)| = |B_1(s_0)| + |B_2(s_0)|.$$

So, given a fixed point $x_0 \in \partial\mathcal{D}$, we have the existence of four distinct points $p_1 = \alpha(s_0)$, $p_2 = \alpha(L/2 - s_0)$, $p_3 = \alpha(L/2 + s_0)$ and $p_4 = \alpha(-s_0)$ ordered counter-clockwise so that

$$|A_1| + |A_2| = |B_1| + |B_2|,$$

where

$$\begin{array}{rcl} B_1 & = & \gamma(p_1, p_2) \\ A_1 & = & \gamma(p_2, p_3) \\ B_2 & = & \gamma(p_3, p_4) \\ A_2 & = & \gamma(p_4, p_1). \end{array}$$

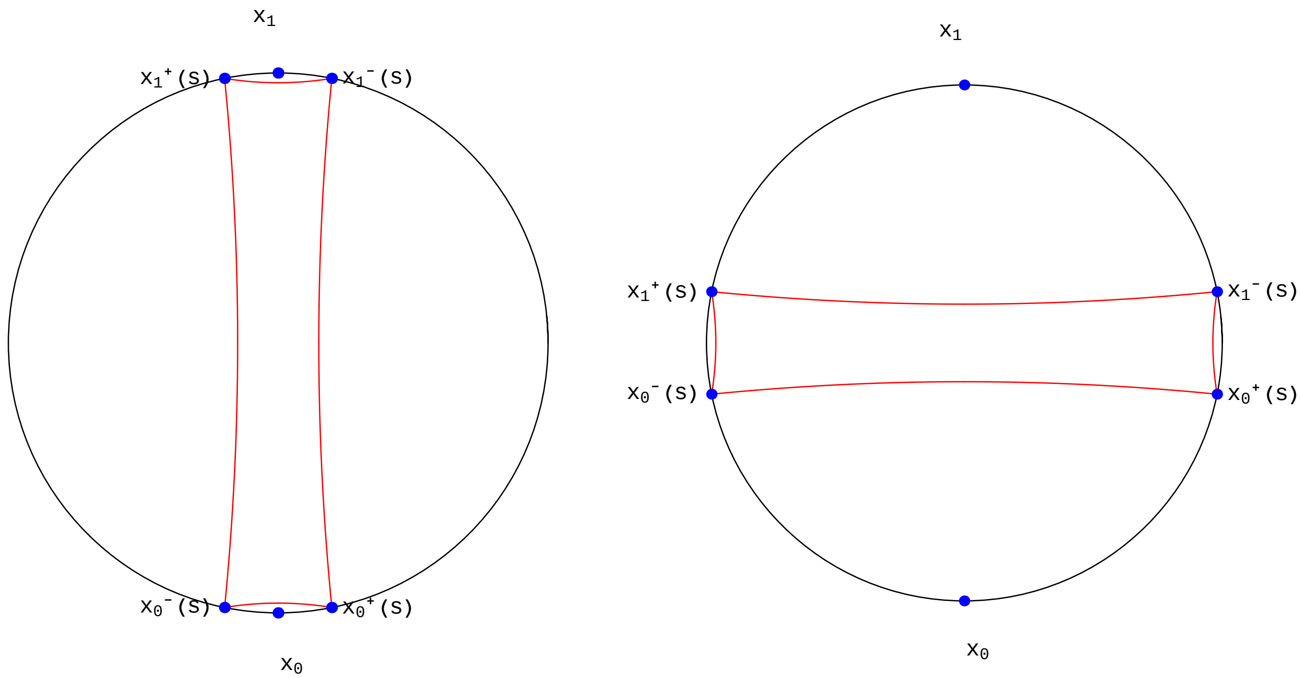


Figure 2: How does the length change?

In analogy with the Euclidean case [3],

Definition 3.1. *Fix a point* $x_0 \in \partial\mathcal{D}$*, let* p_i*,* $i = 1, \ldots, 4$ *be the points constructed above associated to* $x_0 \in \mathcal{D}$*, then* $\Gamma_{x_0} = A_1 \cup B_1 \cup A_2 \cup A_3$ *is called the* **quadrilateral associated to** $x_0 \in \mathcal{D}$ *and it satisfies*

$$|A_1| + |A_2| = |B_1| + |B_2|,$$

where

$$\begin{aligned} B_1 &= \gamma(p_1, p_2) \\ A_1 &= \gamma(p_2, p_3) \\ B_2 &= \gamma(p_3, p_4) \\ A_2 &= \gamma(p_4, p_1). \end{aligned}$$

Moreover, the interior domain D_{x_0} *bounded by* Γ_{x_0} *is the* **square inscribed associated to** $x_0 \in \mathcal{D}$ *(note that* D_{x_0} *is a topological disc), and* B_1 *is called the* **bottom side** *(c.f. Figure 3).*

Second, let us explain how to take the *regular trapezoids*: As above, fix $x_0 \in \partial\mathcal{D}$ (from now on, x_0 will be fixed and we will omit it) and parametrize $\partial\mathcal{D}$ as $\alpha : \mathbb{R}/[0, L) \longrightarrow \partial\mathcal{D}$. Let

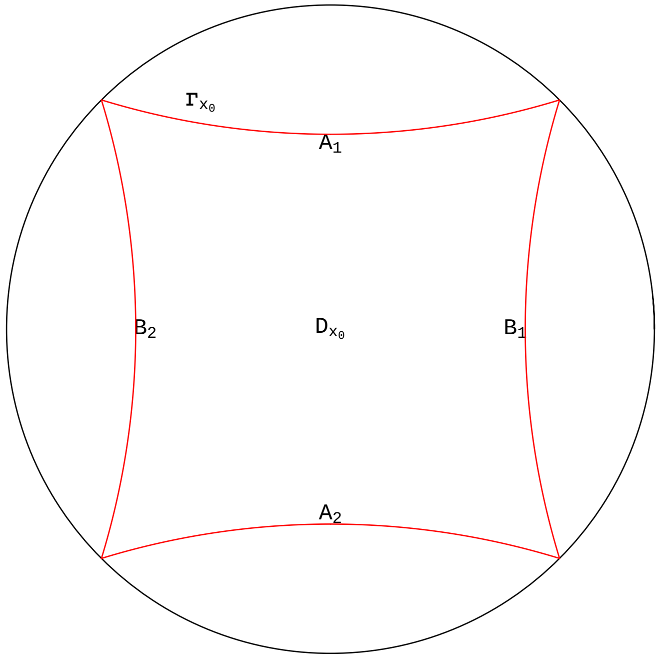


Figure 3: Scherk domain

$0 \leq s_1 < s_2 < L$, or equivalently, two distinct and ordered points $p_i = \alpha(s_i) \in \partial\mathcal{D}$, $i = 1, 2$. The aim is to construct a *trapezoid* in the region bounded by $\gamma(p_1, p_2)$ and $\alpha([s_1, s_2])$. To do so, set $\bar{s} = \frac{s_1+s_2}{2}$, i.e., $\bar{p} = \alpha(\bar{s})$ is the mid-point. Define $\bar{p}^{\pm}(s) = \alpha(\bar{s} \pm s)$ for $0 \leq s \leq \bar{s}$.

Set

$$\begin{aligned} l_1(s) &= \text{Length}\left(\gamma(p_1, \bar{p}^-(s))\right) \\ l_2(s) &= \text{Length}\left(\gamma(\bar{p}^-(s), \bar{p}^+(s))\right) \\ l_3(s) &= \text{Length}\left(\gamma(\bar{p}^+(s), p_2)\right) \\ l_4(s) &= \text{Length}\left(\gamma(p_2, p_1)\right). \end{aligned}$$

Hence, for s close to zero

$$l_1(s) + l_3(s) > l_2(s) + l_4(s)$$

by the Triangle Inequality, and for s close to $\bar{s}$

$$l_1(s) + l_3(s) < l_2(s) + l_4(s),$$

since l_1 and l_3 go to zero and l_4 has positive length (c.f. Figure 4).

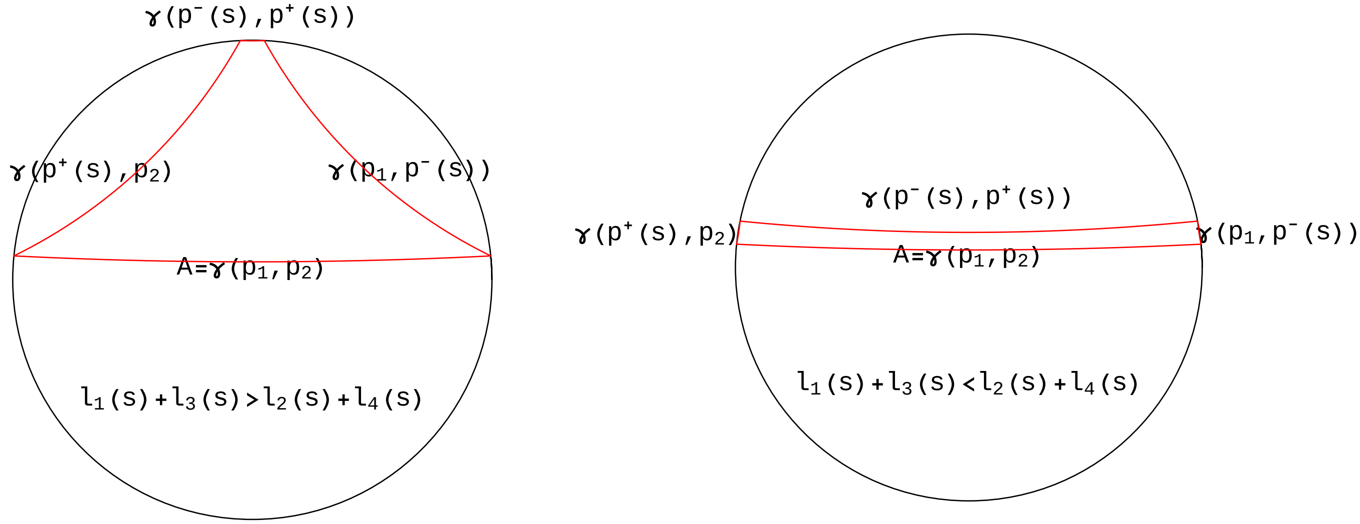


Figure 4: How does the *trapezoid* vary?

Thus, there exists $s_0 \in (0, \bar{s})$ so that

$$l_1(s_0) + l_3(s_0) = l_2(s_0) + l_4(s_0).$$

So, given a fixed point $x_0 \in \partial\mathcal{D}$ and a geodesic arc $A := \gamma(p_1, p_2)$ joining two (distinct and oriented) points in $\partial\mathcal{D}$, we have the existence of two distinct points $p^- = \alpha(\bar{s} - s_0)$ and $p^+ = \alpha(\bar{s} + s_0)$ ordered count-clockwise so that

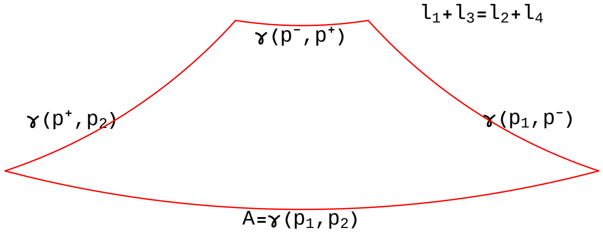
$$l_1 + l_3 = l_2 + l_4,$$

where

$$\begin{aligned} l_1 &= \text{Length}\left(\gamma(p_1, p^-)\right) \\ l_2 &= \text{Length}\left(\gamma(p^-, p^+)\right) \\ l_3 &= \text{Length}\left(\gamma(p^+, p_2)\right) \\ l_4 &= \text{Length}\left(\gamma(p_2, p_1)\right). \end{aligned}$$

Moreover, the domain bounded by $\gamma(p_1, p^-) \cup \gamma(p^-, p^+) \cup \gamma(p^+, p_2) \cup \gamma(p_1, p_2)$ is a topological disc.

Again, in analogy with the Euclidean case,



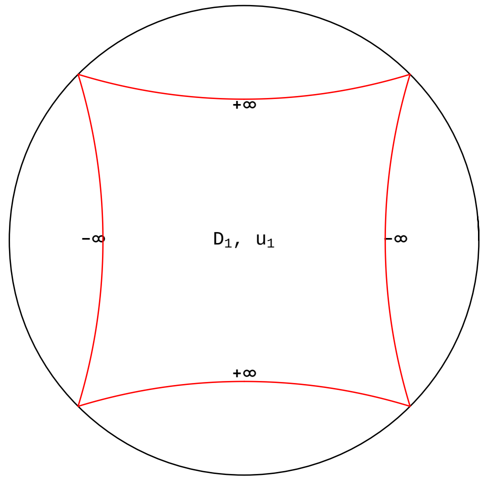


Figure 5: (Left) Regular Trapezoid

Figure 6: (Right) First Scherk domain

Definition 3.2. *$E = \gamma(p_1, p^-) \cup \gamma(p^-, p^+) \cup \gamma(p^+, p_2) \cup \gamma(p_1, p_2)$ is called the* **regular trapezoid associated to the side** *A, here $A = \gamma(p_1, p_2)$ (and, of course, once we have fixed a point $x_0 \in \partial\mathcal{D}$), and $p^{\pm}$ are given by the above construction (c.f. Figure 5).*

Now, we can begin the example. We only highlight the main steps in the construction since, in essence, it is as in [3, Section III].

Fix $x_0 \in \partial\mathcal{D}$ and let D_1 the inscribed quadrilateral associated to x_0 and $\Gamma_1 = \partial D_1$ (see Definition 3.1). We label A_1, B_1, A_2, B_2 the sides of Γ_1 ordered count-clockwise, with B_1 the bottom side. By construction, D_1 is a Scherk domain. One can check this fact using the Triangle Inequality. From Theorem 3.1, there is a minimal graph u_1 in D_1 which is $+\infty$ on the $A_i's$ sides and equals $-\infty$ on the $B_i's$ sides (c.f. Figure 6).

Henceforth, we will attach regular trapezoids (see Definition 3.2) to the sides of the quadrilateral Γ_1 in the following way. Let E_1 the regular trapezoid associated to the side A_1, and E_1' the regular trapezoid associated to the side B_1.

Consider the domain $D_2 = D_1 \cup E_1 \cup E_1'$, $\Gamma_2 = \partial D_2$. This new domain does not satisfy the second condition of Theorem 3.1 ,we only have to consider the inscribed polygon E (c.f. Figure 7).

So, the next step is to perturb D_2 in such a way that it becomes an admissible domain. Let p be the common vertex of E_1 and E_1'. Let a_1 the closed vertex of E_1 to p, and b_1 the closed vertex of E_1' to p (c.f. Figure 8).

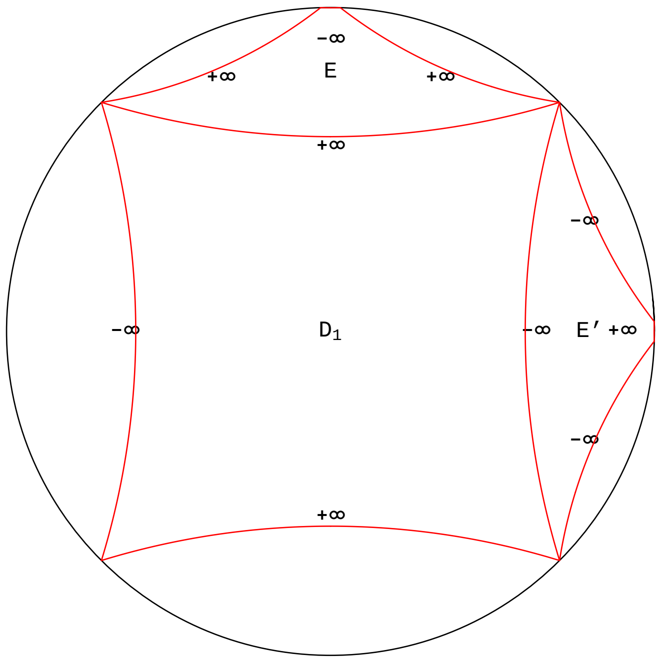


Figure 7: Attaching trapezoids

One moves the vertex a_1 towards b_1 to a nearby point $a_1(\tau)$ on $\partial\mathcal{D}$ (using the parametrization $\alpha : \mathbb{R}/[0, L) \longrightarrow \partial\mathcal{D}$ as we have been done throughout this Section). And then one moves b_1 towards a_1 to a nearby point $b_1(\tau)$ on $\partial\mathcal{D}$.

Let $\Gamma_2(\tau)$ the inscribed polygon obtained by this perturbation, $E_1(\tau)$ and $E_1'(\tau)$ the perturbed regular trapezoids (c.f. Figure 9). Thus, for $\tau > 0$ small, it is clear that:

- $\Gamma_2(\tau)$ satisfies Condition 1 in Theorem 3.1.
- $2a(E_1(\tau)) < |E_1(\tau)|$ and $2b(E_1'(\tau)) < |E_1'(\tau)|$.

Now, we state the following Lemma that establish how we extend the Scherk surface in general.

Lemma 3.1. *Let u be a Scherk graph on a polygonal domain $D_1 = P(A_1, B_1, \ldots, A_k, B_k)$, where the $A_i's$ and $B_i's$ are the (geodesic) sides of ∂D_1 on which u takes values $+\infty$ and $-\infty$ respectively. Let K be a compact set in the interior of D_1. Let $D_2 = P(E_1, E_1', A_2, B_2, \ldots, A_k, B_k)$ be the polygonal domain D_1 to which we attach two regular trapezoids E_1 to the side A_1 and E_1' to the side B_1. Let $E_1(\tau)$ and $E_1'(\tau)$ be the perturbed polygons as above. Then*

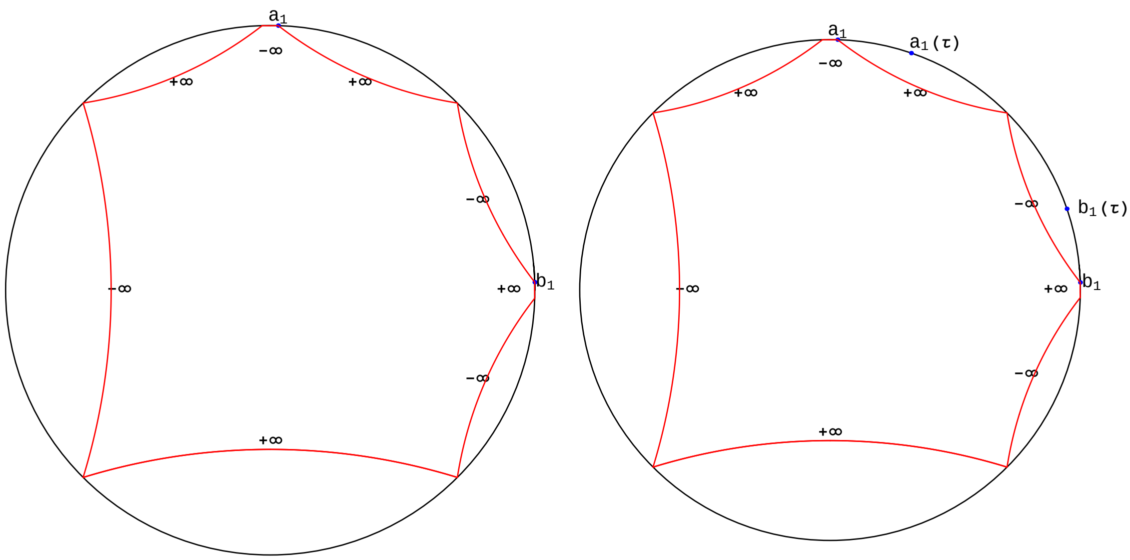


Figure 8: Moving the vertex of the trapezoid

for all $\epsilon > 0$ *there exists* $\bar{\tau} > 0$ *so that, for all* $0 < \tau \leq \bar{\tau}$*,* v *is a Scherk graph on* $P(E_1(\tau), E_1'(\tau), A_2, B_2, \ldots, A_k, B_k)$ *such that*

$$\|u - v\|_{C^2(K)} \leq \epsilon. \tag{3.1}$$

Proof. The proof of this Lemma relies on [3, Section IV] with the obvious differences that we need to use the results for Scherk graphs over a domain in a Hadamard surface stated in [9] and [5]. □

Before we return to the construction, let us explain how we construct a *compact domain associated to any Scherk domain*: Let $D = P(A_1, B_1, \ldots, A_k, B_k)$ be a Scherk domain in $\mathcal{D}$ with vertex $\{v_1, \ldots, v_{2k}\} \in \partial\mathcal{D}$. Let $\beta_{v_i} : [0,1] \longrightarrow \overline{\mathcal{D}}$ denote the radial geodesic starting at $p_0 \in \mathcal{D}$ (the center of the disc $\mathcal{D}$) and ending at $v_i \in \partial\mathcal{D}$. Note that any β_{v_i} can not touch neither a A_i side nor a B_i side expect at the vertex.

Set $r < 1$ and $p_i = \beta_{v_i}(r) \in \mathcal{D}$ for $i = 1, \ldots, 2k$. Consider the polygon

$$P = \bigcup_{i=1}^{2k-1} \gamma(p_i, p_{i+i}) \cup \gamma(p_{2k}, p_1) \subset D,$$

and let K' be the closure of the domain bounded by P, here $\gamma(p_i, p_{i+1})$ is the geodesic arc joining p_i and p_{i+1} in D. Let $\mathcal{D}(p_i, 1-r)$ be geodesic disc centered at p_i of radius $1-r$ for each $i = 1, \ldots, 2k$. Then,

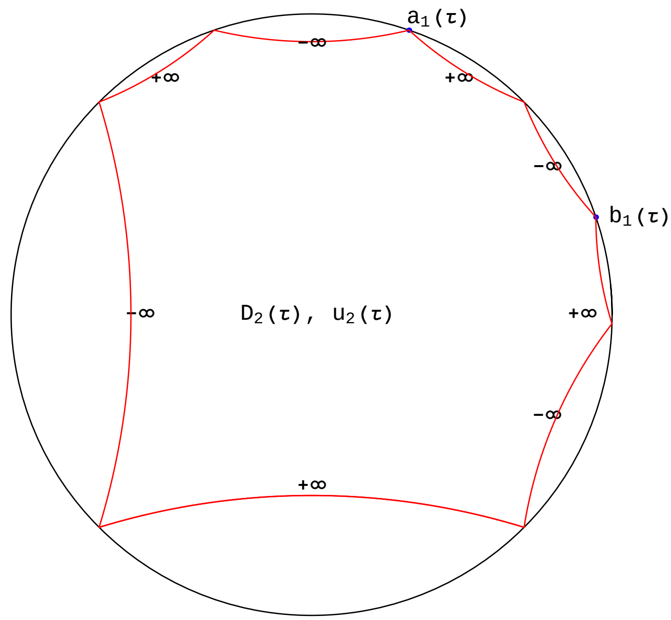


Figure 9: Perturbed Scherk domain

Definition 3.3. *For $r < 1$ close to 1, the* **compact domain associated to the Scherk domain** *D is given by*

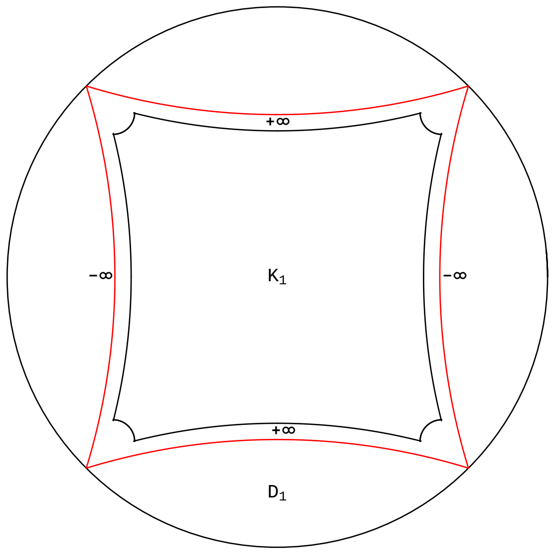
$$K = K' \setminus \bigcup_{i=1}^{2k} \mathcal{D}(p_i, 1-r).$$

Now, we continue with the construction. Let $D_1 = P(A_1, B_1, A_2, B_2)$ be the inscribed square in $\mathcal{D}$ (given in Definition 3.1), and the Scherk graph u_1 on D_1 which is $+\infty$ on the $A_i's$ sides and $-\infty$ on the $B_i's$ sides. Let K_1 be the compact domain associated to D_1 (see Definition 3.3). We choose $r_1 < 1$ close enough to one so that $u_1 > 1$ on the geodesic sides of ∂K_1 closer to the $A_i's$ sides and $u_1 < -1$ on the geodesic sides of ∂K_1 closer to the $B_i's$ sides (cf. Figure 10).

Next, we attach perturbed regular trapezoids to the sides A_1 and B_1, so from Lemma 3.1, for any $\epsilon_2 > 0$ there exists $\tau_2 > 0$ so that $D_2(\tau) = D_1 \cup E_1(\tau) \cup E_1'(\tau)$ is a Scherk domain and $u_2(\tau)$, the Scherk graph defined on $D_2(\tau)$, satisfy

$$\|u_1 - u_2(\tau)\|_{C^2(K_1)} \leq \epsilon_2,$$

for all $0 < \tau \leq \tau_2$. Moreover, we can choose $u_2(\tau)$ so that $u_1(p_0) = u_2(\tau)(p_0)$ (here p_0 is the center of $\mathcal{D}$). Then, choose $\epsilon_2 > 0$ so that $u_2(\tau) > 1$ on the geodesic sides of ∂K_1 closer to the



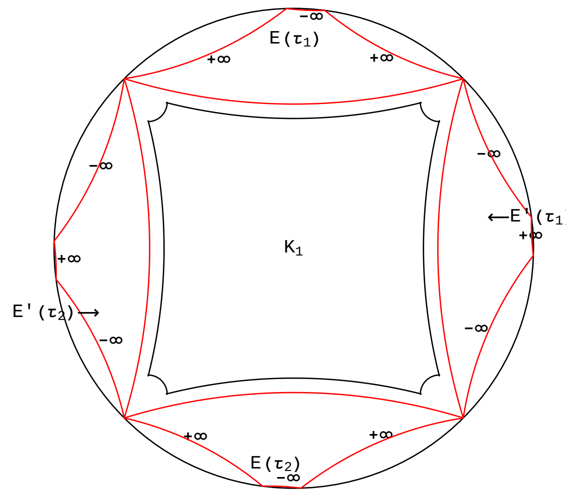


Figure 10: (Left) Compact domain associated to the inscribed quadrilateral

Figure 11: (Right) Attaching perturbed regular trapezoids

$A_i's$ sides and $u_2(\tau) < -1$ on the geodesic sides of ∂K_1 closer to the $B_i's$ sides.

Let $K_2(\tau)$ be the compact domain associated to the Scherk domain $D_2(\tau)$. Choose $r_2 < 1$ close enough to one (in the definition of $K_2(\tau)$ given by Definition 3.3) so that, for $0 < \tau \leq \tau_2$, $u_2(\tau) > 2$ on those geodesic sides of $\partial K_2(\tau)$ parallel to the sides of $D_2(\tau)$ where $u_2(\tau) = +\infty$, and $u_2(\tau) < -2$ on the sides of $\partial K_2(\tau)$ parallel to sides of $D_2(\tau)$ where $u_2(\tau) = -\infty$ (cf. Figure 12).

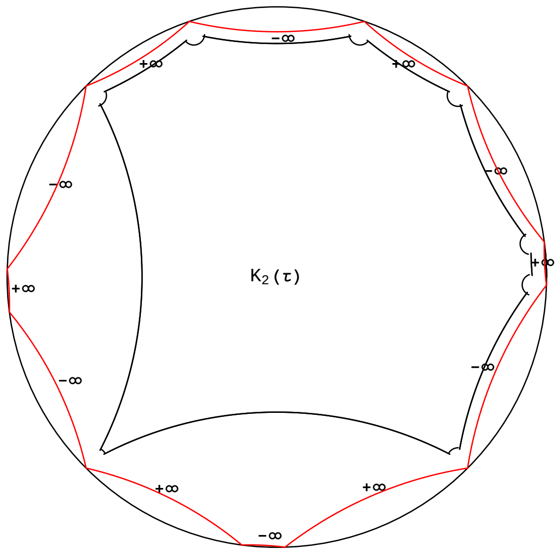
Continue by constructing the Scherk domain $D_3(\tau)$ by attaching perturbed regular trapezoids (as above) to the sides A_2 and B_2 of D_1. We know, for $\epsilon_3 > 0$, that there exist $\tau_3 > 0$ so that if $0 < \tau \leq \tau_3$ then the Scherk graph $u_3(\tau)$ exists, $u_3(\tau)(p_0) = u_1(p_0)$ and

$$\|u_3(\tau) - u_2(\tau)\|_{C^2(K_2(\tau))} \leq \epsilon_3.$$

Moreover, choose $\epsilon_3 > 0$ so that $u_3(\tau) > 3$ on the geodesic sides of $\partial K_2(\tau)$ closer to the $A_i's$ sides and $u_3(\tau) < -3$ on the geodesic sides of $\partial K_2(\tau)$ closer to the $B_i's$ sides (cf. Figure 13).

Now choose $\epsilon_n \longrightarrow 0$, $\tau_n \longrightarrow 0$, $K_n(\tau_n)$ so that $K_n(\tau_n) \subset K_{n+1}(\tau_{n+1})$, $\bigcup_n K_n(\tau_n) = \mathcal{D}$. Then the $u_n(\tau_n)$ converge to a graph u on $\mathcal{D}$.

To see u has the desired properties, we refer the reader to [3, pages 13 and 14] with the only difference that we need to use now Theorem 2.1.



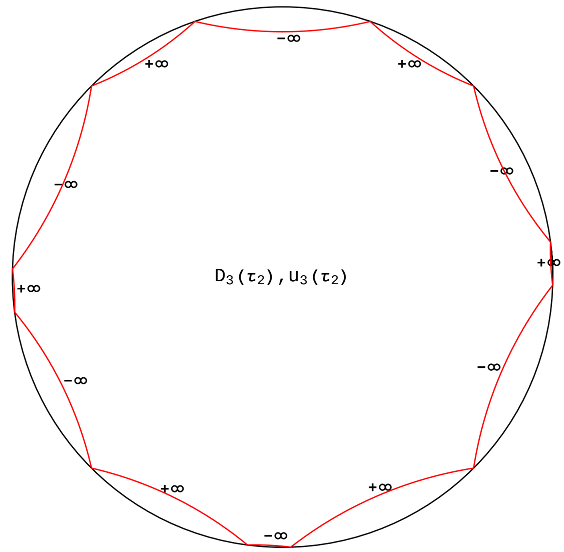


Figure 12: (Left) Compact domain associated to $D_2(\tau)$

Figure 13: (Right) Choosing $u_3(\tau)$

Remark 3.1. *The above construction can be carried out in a more general situation. Actually, if we ask that*

- *The geodesic disc $\mathcal{D}$ has strictly convex boundary.*
- *There is a unique minimizing geodesic joining any two points of the disc.*

Then, we can extend the above example.

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Fatou's Theorem and minimal graphs

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Abstract

In this paper we extend a recent result of Collin-Rosenberg (*a solution to the minimal surface equation in the Euclidean disc has radial limits almost everywhere*) to a large class of differential operators in Divergence form. Moreover, we construct an example (in the spirit of [3]) of a minimal graph in $\mathbb{M}^2 \times \mathbb{R}$, where $\mathbb{M}^2$ is a Hadamard surface, over a geodesic disc which has finite radial limits in a mesure zero set.

1 Introduction

It is well known that a bounded harmonic function u defined on the Euclidean disc D has radial limits almost everywhere (Fatou's Theorem [4]). Moreover, the radial limits can not be plus infinity for a positive measure set. For fixed $\theta \in \mathbb{S}^{n-1}$, the radial limit $u(\theta)$ (if it exists) is defined as

$$u(\theta) = \lim_{r \to 1} u(r, \theta),$$

where we paramatrize the Euclidean disc in polar coordinates $(r, \theta) \in [0, 1) \times \mathbb{S}^1$.

In 1965, J. Nitsche [8] asked if a Fatou Theorem is valid for the minimal surface equation, i.e., *does a solution for the minimal surface equation in the Euclidean disc have radial limits almost everywhere?* This question has been solved recently by P. Collin and H. Rosenberg [3]. Moreover, in the same paper [8], J. Nitsche asked: *what is the largest set of θ for which a minimal graph on D may not have radial limits?* Again, this question was solved in [3] if

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one allows infinite radial limits. That is, they construct an example of a minimal graph in the Euclidean disc with finite radial limits only on a set of measure zero. In this example, the $+\infty$ radial limits (resp. $-\infty$) are taken on a set of measure π (resp. π).

The aim of this paper is to extend both results. In Section 2, we extend Collin-Rosenberg's Theorem to a large class of differential operators in divergence form (see Theorem 2.1). We show this applies to minimal graph sections of Heisenberg space. In Section 3, we construct an example of a minimal graph in $\mathbb{M}^2 \times \mathbb{R}$ over a geodesic disk $\mathcal{D} \subset \mathbb{M}^2$ ($\mathbb{M}^2$ is a Hadamard surface) for which the finite radial limits are of measure zero. Also, the $+\infty$ radial limits (resp. $-\infty$) are taken on a set of measure π (resp. π).

2 Fatou's Theorem

Henceforth $(\mathbb{B}, g)$ denotes the $n-$dimensional unit open ball, i.e,

$$\mathbb{B} = \left\{(r,\theta)\,;\, 0 \leq r < 1,\, \theta \in \mathbb{S}^{n-1}\right\},$$

in polar coordinates with respect to g, g a C^2-Riemannian metric on $\mathbb{B}$. Define $G := G(r,\theta) = \sqrt{\det(g)}$. Moreover, we denote by ∇ the Levi-Civita connection associated to g and by div_g its associated divergence operator. Also, $L^1(\mathbb{B})$ denotes the set of integrable functions on $(\mathbb{B}, g)$.

Set $u \in C^2(\mathbb{B})-$function and X_u be a $C^1(\mathbb{B})-$vector field so that its coordinates depend on u, its first derivatives and $C^1(\mathbb{B})-$functions.

For fixed $\theta \in \mathbb{S}^{n-1}$, the radial limit (if it exists) $u(\theta)$ is defined as

$$u(\theta) = \lim_{r\to 1} u(r,\theta).$$

Theorem 2.1. *Let $(\mathbb{B}, g, G, u, X_u)$ be as above. Assume that*

a) $\alpha \leq G(r,\theta) \leq \beta$ for all $(r,\theta) \in [1/2, 1) \times \mathbb{S}^{n-1}$, α and β positive constants.

b) $|X_u| \leq M$ on $\mathbb{B}$, i.e., X_u is bounded on $\mathbb{B}$.

c) $g(\nabla u, X_u) \geq \delta\,|\nabla u| + h$, where δ is a positive constant and $|h| \in L^1(\mathbb{B})$.

Let $|f| \in L^1(\mathbb{B})$. If u is a solution of

$$\mathrm{div}_g(X_u) \geq \ (\ \textit{or} \ \ \leq)\ f \ \ \textit{on}\ \mathbb{B},$$

then u has radial limits almost everywhere.

Proof. First, let us prove the case

$$\mathrm{div}_g(X_u) \geq \ f.$$

For $r < 1$ fixed, set $\mathbb{B}(r)$ the $n-$dimensional open ball of radius r. Let $\eta : \mathbb{R} \longrightarrow (0, +1)$ be a smooth function so that $0 < \eta'(x) < 1$ for all $x \in \mathbb{R}$. Define $\psi := \eta \circ u$.

On the one hand, by direct computations and *item c)*, we have

$$\begin{aligned}\operatorname{div}_g(\psi\, X_u) &= \psi \operatorname{div}_g(X_u) + g(\nabla\psi, X_u) \geq \psi\, f + \eta'\, g(\nabla u, X_u) \\ &\geq \psi\, f + \eta'\; (\delta\, |\nabla u| + h) = \delta\, \eta'\, |\nabla u| + (\psi\, f + \eta'\, h) \\ &= \delta\, |\nabla\psi| + (\psi\, f + \eta'\, h)\,,\end{aligned}$$

thus

$$\int_{\mathbb{B}(r)} \operatorname{div}_g(\psi\, X_u) \geq \delta \int_{\mathbb{B}(r)} |\nabla\psi| + C \tag{2.1}$$

where C is some constant. This follows since $|h|$ and $|f|$ are L^1-functions on $\mathbb{B}$.

On the other hand, by Stokes' Theorem and *items a)* and $b)$, we obtain for $r < 1$ fixed

$$\begin{aligned}\int_{\mathbb{B}(r)} \operatorname{div}_g(\psi\, X_u) &= \int_{\partial\mathbb{B}(r)} \psi\, g(X_u, \upsilon) \leq \int_{\partial\mathbb{B}(r)} M \\ &= M \int_{\theta\in\mathbb{S}^{n-1}} G(r,\theta) d\theta \leq M\, \beta \int_{\theta\in\mathbb{S}^{n-1}} \\ &= M\, \beta\, \omega_{n-1},\end{aligned} \tag{2.2}$$

where υ is the outer conormal to $\partial\mathbb{B}(r)$ and ω_{n-1} is the volume of $\mathbb{S}^{n-1}$.

So, from (2.1), (2.2) and letting r go to one, we conclude that $|\nabla\psi|$ is integrable in $\mathbb{B}$, i.e.,

$$\int_{\mathbb{B}} |\nabla\psi| < +\infty \tag{2.3}$$

Since $\dfrac{\partial\psi}{\partial r} \leq |\nabla\psi|$, we have from Fubini's Theorem and (2.3)

$$\int_{\mathbb{B}} \frac{\partial\psi}{\partial r} = \int_{\theta\in\mathbb{S}^{n-1}} \left(\int_0^1 \frac{\partial\psi}{\partial r} G(r,\theta)\, dr \right) d\theta < \infty.$$

Thus, as $G(r,\theta)$ is bounded below by a positive constant, for $r > 1/2$ and almost all $\theta \in \mathbb{S}^{n-1}$,

$$\lim_{r\to 1} \psi(r,\theta) - \psi(0,0) = \int_0^1 \frac{\partial\psi}{\partial r}(r,\theta) dr < \infty,$$

that is, ψ has radial limits almost everywhere. Since $\psi = \eta \circ u$, we conclude u has radial limits almost everywhere (which may be $\pm\infty$).

For

$$\operatorname{div}_g(X_u) \leq f,$$

we just have to follow the above proof by changing $\eta : \mathbb{R} \longrightarrow (-1, 0)$ so that $0 < \eta'(x) < 1$ for all $x \in \mathbb{R}$. $\square$

2.1 Applications

Moreover, we will see now how Theorem 2.1 applies to get radial limits almost everywhere for minimal graphs in ambient spaces besides $\mathbb{R}^3$. We work here in Heisenberg space, but it is not hard to check that we could work with minimal graphs in a more general submersion (see [7]).

First, we need to recall some definitions in Heisenberg space (see [1]). The Heisenberg spaces are $\mathbb{R}^3$ endowed with a one parameter family of metrics indexed by bundle curvature by a real parameter $\tau \neq 0$. When we say the *Heisenberg space*, we mean $\tau = 1/2$, and we denote it by $\mathcal{H}$.

In global exponential coordinates, $\mathcal{H}$ is $\mathbb{R}^3$ endowed with the metric

$$g = (dx^2 + dy^2) + (\frac{1}{2}(ydx - xdy) + dz)^2.$$

The Heisenberg space is a Riemannian submersion $\pi : \mathcal{H} \longrightarrow \mathbb{R}$ over the standard flat Euclidean plane $\mathbb{R}^2$ whose fibers are the vertical lines, i.e., they are the trajectories of a unit Killing vector field and hence geodesics.

Let $S_0 \subset \mathcal{H}$ be the surface whose points satisfy $z = 0$. Let $D \subset \mathbb{R}^2$ be the unit disc. Henceforth, we identify domains in $\mathbb{R}^2$ with its lift to S_0. The Killing graph of a function $u \in C^2(D)$ is the surface

$$\Sigma = \{(x, y, u(x, y))\,;\,(x, y) \in D\}\,.$$

Moreover, the minimal graph equation is

$$\mathrm{div}_{\mathbb{R}^2}(X_u) = 0,$$

here $\mathrm{div}_{\mathbb{R}^2}$ stands for the divergence operator in $\mathbb{R}^2$ with the Euclidean metric $\langle , \rangle$, and

$$X_u := \frac{\alpha}{W}\partial_x + \frac{\beta}{W}\partial_y,$$

where

$$\alpha := \frac{y}{2} + u_x, \quad \beta := -\frac{x}{2} + u_y,$$

and

$$W^2 = 1 + \alpha^2 + \beta^2.$$

Thus, for verifying u has radial limits almost everywhere (which may be $\pm\infty$), we have to check conditions $a)$, $b)$ and $c)$. Item $a)$ is immediate since we are working with the Euclidean metric.

Item $b)$ follows from

$$|X_u|^2 = \frac{\alpha^2 + \beta^2}{1 + \alpha^2 + \beta^2} \leq 1.$$

Now, we need to check Item $c)$. On one hand, using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned}
W^2 = 1+\alpha^2+\beta^2 &= 1+u_x^2+u_y^2+(yu_x-xu_y)+\frac{x^2+y^2}{4} \\
&= 1+|\nabla u|^2+\langle \nabla u, (-y,x)\rangle+\frac{x^2+y^2}{4} \\
&\geq 1+|\nabla u|^2-|\nabla u||(-y,x)|+\frac{x^2+y^2}{4} \\
&= 1+|\nabla u|^2-r|\nabla u|+\frac{r^2}{4}
\end{aligned}$$

thus,

$$W \geq \sqrt{1+\left(|\nabla u|-\frac{r}{2}\right)^2} \geq \left||\nabla u|-\frac{r}{2}\right|.$$

We need a lower bound for W in terms of $|\nabla u|$. To do so, we distinguish two cases:

Case $|\nabla u| \leq 5/4$**:** Since

$$1-r\,|\nabla u|+\frac{r^2}{4} \geq 1-\frac{5r}{4}+\frac{r^2}{4} \geq 0 \text{ for all } r \leq 1,$$

we obtain

$$W \geq \sqrt{|\nabla u|^2+1-r\,|\nabla u|+\frac{r^2}{4}} \geq |\nabla u|\,.$$

Case $|\nabla u| > 5/4$: We already know that

$$W \geq \left||\nabla u|-\frac{r}{2}\right|,$$

thus, for $|\nabla u| > 5/4$, it is easy to see that

$$\left||\nabla u|-\frac{r}{2}\right| \geq \frac{3}{10}\,|\nabla u| \text{ for all } r \leq 1.$$

So, in any case, for $\delta = 3/10 > 0$

$$W \geq \delta\,|\nabla u|\,. \tag{2.4}$$

On the other hand,

$$\begin{aligned}
\langle \nabla u, X_u\rangle &= \frac{u_x^2+u_y^2+\frac{1}{2}(yu_x-xu_y)}{W} \\
&= \frac{1+u_x^2+u_y^2+(yu_x-xu_y)+\frac{x^2+y^2}{4}}{W}-\frac{1+\frac{1}{2}(yu_x-xu_y)+\frac{x^2+y^2}{4}}{W} \\
&= \frac{W^2}{W}+h = W+h \geq \delta\,|\nabla u|+h,
\end{aligned}$$

where we have used (2.4) and h denotes the bounded function

$$h = -\frac{1 + \frac{1}{2}(yu_x - xu_y) + \frac{x^2+y^2}{4}}{\sqrt{1 + u_x^2 + u_y^2 + (yu_x - xu_y) + \frac{x^2+y^2}{4}}},$$

that is, Item $c)$ is satisfied. So,

Corollary 2.1. *A solution for the minimal surface equation in the Heisenberg space defined over a disc has radial limits almost everywhere (which may be $\pm\infty$).*

3 An example in a Hadamard surface

The aim of this Section is to construct an example of a minimal graph in $\mathbb{M}^2 \times \mathbb{R}$ over a geodesic disk $\mathcal{D} \subset \mathbb{M}^2$ ($\mathbb{M}^2$ is a Hadamard surface) for which the finite radial limits are of measure zero.

We need to recall preliminary facts about graphs over a Hadamard surface (see [5] for details). Henceforth, $\mathbb{M}^2$ denotes a simply connected with Gauss curvature bounded above by a negative constant, i.e., $K_{\mathbb{M}^2} \leq c < 0$.

Let $p_0 \in \mathbb{M}^2$ and $\mathcal{D}$ be the the geodesic disk in $\mathbb{M}^2$ centered at p_0 of radius one. Re-scaling in the metric, we can assume that

$$\max\left\{K_{\mathbb{M}^2}(p)\,;\ p \in \overline{\mathcal{D}}\right\} = -1.$$

From the Hessian Comparison Theorem (see e.g. [6]), $\partial\mathcal{D}$ bounds a strictly convex domain. We assume that $\partial\mathcal{D}$ is smooth, otherwise we can work in a smaller disc. We identify $\partial\mathcal{D} = \mathbb{S}^1$ and orient it counter-clockwise.

We say that Γ is an *admissible polygon* in $\mathcal{D}$ if Γ is a Jordan curve in $\overline{\mathcal{D}}$ which is a geodesic polygon with an even number of sides and all the vertices in $\partial\mathcal{D}$. We denote by $A_1, B_1, \ldots, A_k, B_k$ the sides of Γ which are oriented counter-clockwise. Recall that any two sides can not intersect in $\mathcal{D}$. Set D the domain in $\mathcal{D}$ bounded by Γ. By $|A_i|$ (resp. $|B_j|$), we denote the length of such a geodesic arc.

Theorem 3.1 ([9])**.** *Let $\Gamma \subset \mathbb{M}^2$ be a compact polygon with an even number of geodesic sides $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$, in that order, and denote by D the domain with $\partial D = \Gamma$. The necessary and sufficient conditions for the existence of a minimal graph u on D, taking values $+\infty$ on each A_i, and $-\infty$ on each B_j, are the two following conditions:*

1. $\sum_{i=1}^{n} |A_i| = \sum_{i=1}^{n} |B_i|$,
2. *for each inscribed polygon P in D (the vertices of P are among the vertices of Γ) $P \neq D$, one has the two inequalities:*

$$2a(P) < |P| \text{ and } 2b(P) < |P|.$$

Here $a(P) = \sum_{A_j \in P} |A_j|$, $b(P) = \sum_{B_j \in P} |B_j|$ and $|P|$ is the perimeter of P.

The construction of this example follows the steps in [3, Section III], but here we have to be more careful in the choice of the first *inscribed square* and the *trapezoids*. We need to choose them as *symmetric* as possible.

Let us first explain how we take the *inscribed square*: Let $L = \mathrm{length}(\partial\mathcal{D})$ and $\gamma(x_0, x_1)$ be the geodesic arc in $\mathcal{D}$ joining $x_0, x_1 \in \partial\mathcal{D}$. Fix $x_0 \in \partial\mathcal{D}$ and let $\alpha : \mathbb{R}/[0, L) \longrightarrow \partial\mathcal{D}$ an arc-length parametrization of $\partial\mathcal{D}$ (oriented count-clockwise). Set $x_1 = \alpha(L/2)$. Consider $x_0^{\pm}(s) = \alpha(\pm s)$ and $x_1^{\pm}(s) = \alpha(L/2 \pm s)$ for $0 \leq s \leq L/2$ (c.f. Figure 1), and denote

$$\begin{aligned}
B_1(s) &= \gamma(x_0^+(s), x_1^-(s)) \\
A_1(s) &= \gamma(x_1^-(s), x_1^+(s)) \\
B_2(s) &= \gamma(x_1^+(s), x_0^-(s)) \\
A_2(s) &= \gamma(x_0^-(s), x_1^+(s)).
\end{aligned}$$

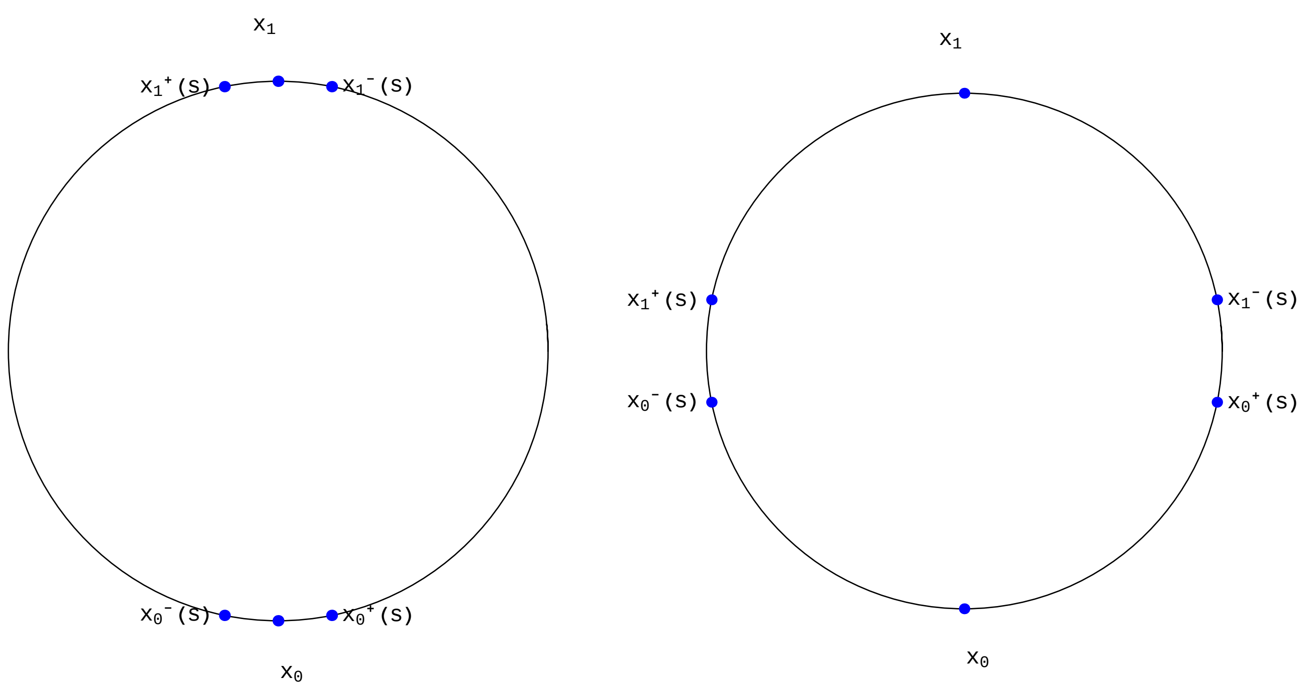


Figure 1: We move the points along $\partial\mathcal{D}$

Hence (c.f. Figure 2),

$$\begin{array}{rcll} |A_1(s)| + |A_2(s)| & > & |B_1(s)| + |B_2(s)| & \text{for } s \text{ close to } 0. \\ |A_1(s)| + |A_2(s)| & < & |B_1(s)| + |B_2(s)| & \text{for } s \text{ close to } L/2. \end{array}$$

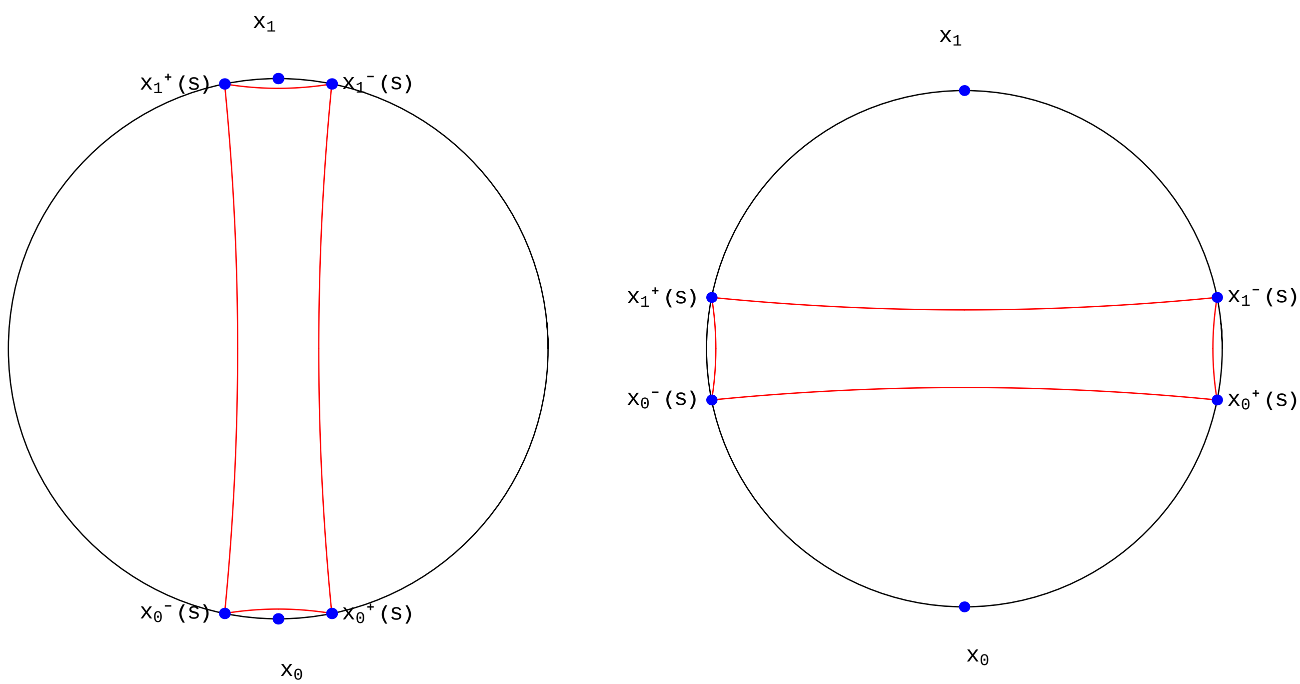


Figure 2: How does the length change?

Thus, there exist $s_0 \in (0, L/2)$ so that

$$|A_1(s_0)| + |A_2(s_0)| = |B_1(s_0)| + |B_2(s_0)|.$$

So, given a fixed point $x_0 \in \partial\mathcal{D}$, we have the existence of four distinct points $p_1 = \alpha(s_0)$, $p_2 = \alpha(L/2 - s_0)$, $p_3 = \alpha(L/2 + s_0)$ and $p_4 = \alpha(-s_0)$ ordered counter-clockwise so that

$$|A_1| + |A_2| = |B_1| + |B_2|,$$

where

$$\begin{array}{rcl} B_1 & = & \gamma(p_1, p_2) \\ A_1 & = & \gamma(p_2, p_3) \\ B_2 & = & \gamma(p_3, p_4) \\ A_2 & = & \gamma(p_4, p_1). \end{array}$$

In analogy with the Euclidean case [3],

Definition 3.1. *Fix a point* $x_0 \in \partial\mathcal{D}$*, let* p_i*,* $i = 1, \ldots, 4$ *be the points constructed above associated to* $x_0 \in \mathcal{D}$*, then* $\Gamma_{x_0} = A_1 \cup B_1 \cup A_2 \cup A_3$ *is called the* **quadrilateral associated to** $x_0 \in \mathcal{D}$ *and it satisfies*

$$|A_1| + |A_2| = |B_1| + |B_2|,$$

where

$$\begin{aligned} B_1 &= \gamma(p_1, p_2) \\ A_1 &= \gamma(p_2, p_3) \\ B_2 &= \gamma(p_3, p_4) \\ A_2 &= \gamma(p_4, p_1). \end{aligned}$$

Moreover, the interior domain D_{x_0} *bounded by* Γ_{x_0} *is the* **square inscribed associated to** $x_0 \in \mathcal{D}$ *(note that* D_{x_0} *is a topological disc), and* B_1 *is called the* **bottom side** *(c.f. Figure 3).*

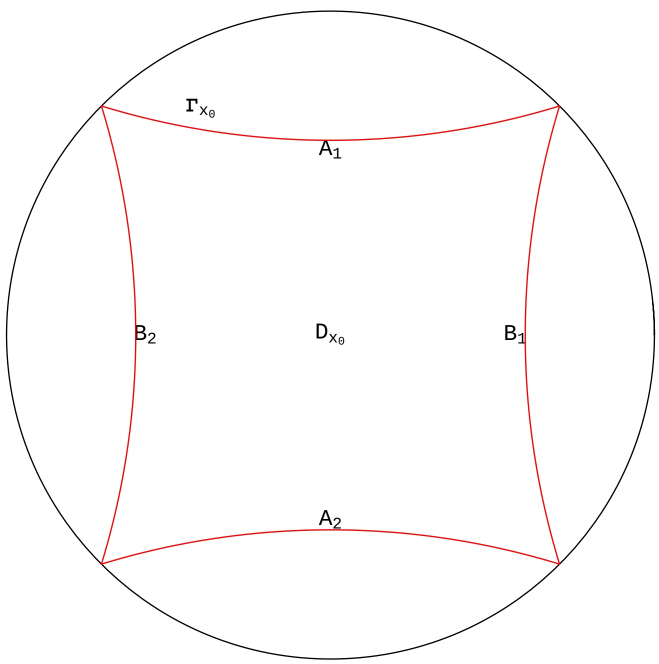


Figure 3: Scherk domain

Second, let us explain how to take the *regular trapezoids*: As above, fix $x_0 \in \partial\mathcal{D}$ (from now on, x_0 will be fixed and we will omit it) and parametrize $\partial\mathcal{D}$ as $\alpha : \mathbb{R}/[0, L) \longrightarrow \partial\mathcal{D}$. Let

$0 \leq s_1 < s_2 < L$, or equivalently, two distinct and ordered points $p_i = \alpha(s_i) \in \partial\mathcal{D}$, $i = 1, 2$. The aim is to construct a *trapezoid* in the region bounded by $\gamma(p_1, p_2)$ and $\alpha([s_1, s_2])$. To do so, set $\bar{s} = \frac{s_1+s_2}{2}$, i.e., $\bar{p} = \alpha(\bar{s})$ is the mid-point. Define $\bar{p}^{\pm}(s) = \alpha(\bar{s} \pm s)$ for $0 \leq s \leq \bar{s}$.

Set

$$
\begin{aligned}
l_1(s) &= \text{Length}\left(\gamma(p_1, \bar{p}^-(s))\right) \\
l_2(s) &= \text{Length}\left(\gamma(\bar{p}^-(s), \bar{p}^+(s))\right) \\
l_3(s) &= \text{Length}\left(\gamma(\bar{p}^+(s), p_2)\right) \\
l_4(s) &= \text{Length}\left(\gamma(p_2, p_1)\right).
\end{aligned}
$$

Hence, for s close to zero

$$l_1(s) + l_3(s) > l_2(s) + l_4(s)$$

by the Triangle Inequality, and for s close to $\bar{s}$

$$l_1(s) + l_3(s) < l_2(s) + l_4(s),$$

since l_1 and l_3 go to zero and l_4 has positive length (c.f. Figure 4).

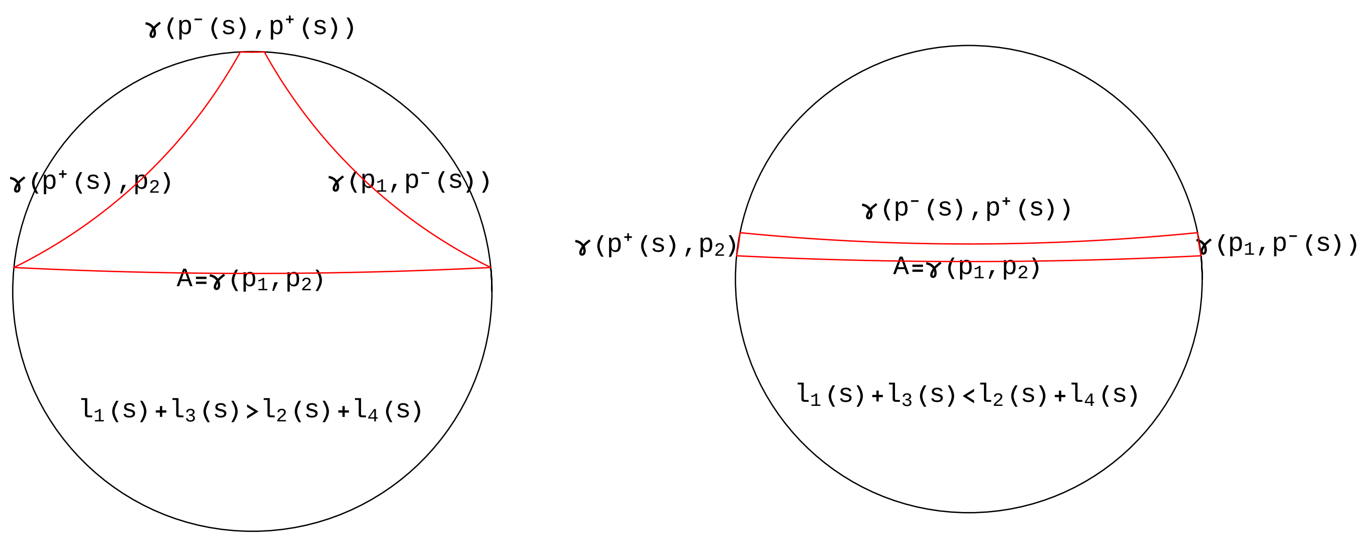


Figure 4: How does the *trapezoid* vary?

Thus, there exists $s_0 \in (0, \bar{s})$ so that

$$l_1(s_0) + l_3(s_0) = l_2(s_0) + l_4(s_0).$$

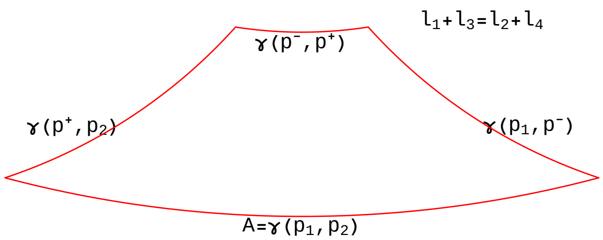
So, given a fixed point $x_0 \in \partial\mathcal{D}$ and a geodesic arc $A := \gamma(p_1, p_2)$ joining two (distinct and oriented) points in $\partial\mathcal{D}$, we have the existence of two distinct points $p^- = \alpha(\bar{s} - s_0)$ and $p^+ = \alpha(\bar{s} + s_0)$ ordered count-clockwise so that

$$l_1 + l_3 = l_2 + l_4,$$

where

$$\begin{aligned} l_1 &= \text{Length}\left(\gamma(p_1, p^-)\right) \\ l_2 &= \text{Length}\left(\gamma(p^-, p^+)\right) \\ l_3 &= \text{Length}\left(\gamma(p^+, p_2)\right) \\ l_4 &= \text{Length}\left(\gamma(p_2, p_1)\right). \end{aligned}$$

Moreover, the domain bounded by $\gamma(p_1, p^-) \cup \gamma(p^-, p^+) \cup \gamma(p^+, p_2) \cup \gamma(p_1, p_2)$ is a topological disc.



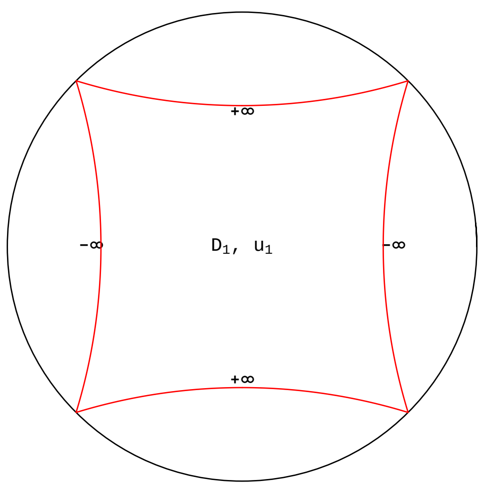


Figure 5: (Left) Regular Trapezoid

Figure 6: (Right) First Scherk domain

Again, in analogy with the Euclidean case,

Definition 3.2. *$E = \gamma(p_1, p^-) \cup \gamma(p^-, p^+) \cup \gamma(p^+, p_2) \cup \gamma(p_1, p_2)$ is called the* **regular trapezoid associated to the side** *A, here $A = \gamma(p_1, p_2)$ (and, of course, once we have fixed a point $x_0 \in \partial\mathcal{D}$), and $p^\pm$ are given by the above construction (c.f. Figure 5).*

Now, we can begin the example. We only highlight the main steps in the construction since, in essence, it is as in [3, Section III].

Fix $x_0 \in \partial\mathcal{D}$ and let D_1 the inscribed quadrilateral associated to x_0 and $\Gamma_1 = \partial D_1$ (see Definition 3.1). We label A_1, B_1, A_2, B_2 the sides of Γ_1 ordered count-clockwise, with B_1 the bottom side. By construction, D_1 is a Scherk domain. One can check this fact using the Triangle Inequality. From Theorem 3.1, there is a minimal graph u_1 in D_1 which is $+\infty$ on the $A_i's$ sides and equals $-\infty$ on the $B_i's$ sides (c.f. Figure 6).

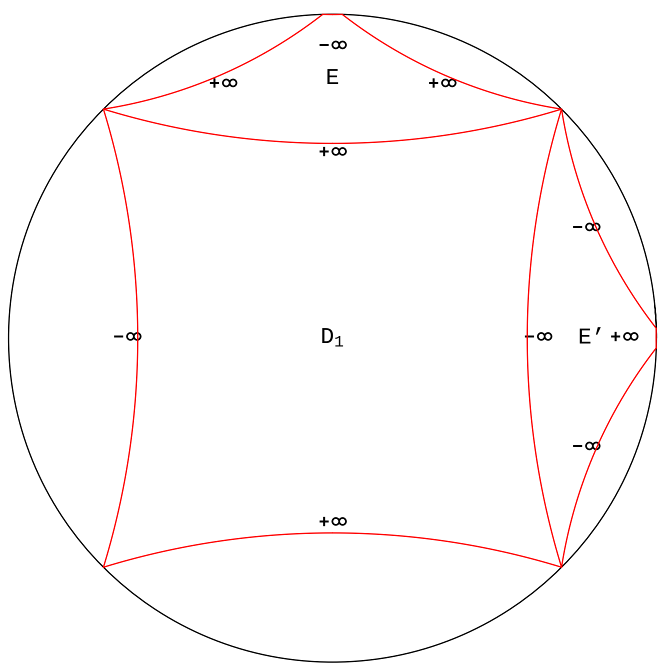


Figure 7: Attaching trapezoids

Henceforth, we will attach regular trapezoids (see Definition 3.2) to the sides of the quadrilateral Γ_1 in the following way. Let E_1 the regular trapezoid associated to the side A_1, and E_1' the regular trapezoid associated to the side B_1.

Consider the domain $D_2 = D_1 \cup E_1 \cup E_1'$, $\Gamma_2 = \partial D_2$. This new domain does not satisfy the second condition of Theorem 3.1 ,we only have to consider the inscribed polygon E (c.f. Figure 7).

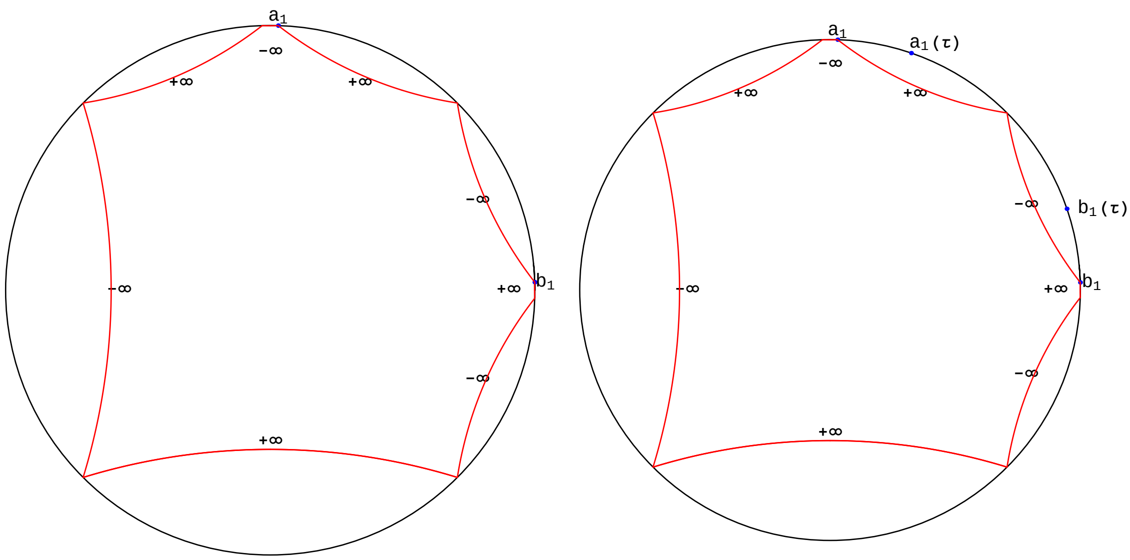


Figure 8: Moving the vertex of the trapezoid

So, the next step is to perturb D_2 in such a way that it becomes an admissible domain. Let p be the common vertex of E_1 and E_1'. Let a_1 the closed vertex of E_1 to p, and b_1 the closed vertex of E_1' to p (c.f. Figure 8).

One moves the vertex a_1 towards b_1 to a nearby point $a_1(\tau)$ on $\partial\mathcal{D}$ (using the parametrization $\alpha : \mathbb{R}/[0, L) \longrightarrow \partial\mathcal{D}$ as we have been done throughout this Section). And then one moves b_1 towards a_1 to a nearby point $b_1(\tau)$ on $\partial\mathcal{D}$.

Let $\Gamma_2(\tau)$ the inscribed polygon obtained by this perturbation, $E_1(\tau)$ and $E_1'(\tau)$ the perturbed regular trapezoids (c.f. Figure 9). Thus, for $\tau > 0$ small, it is clear that:

- $\Gamma_2(\tau)$ satisfies Condition 1 in Theorem 3.1.
- $2a(E_1(\tau)) < |E_1(\tau)|$ and $2b(E_1'(\tau)) < |E_1'(\tau)|$.

Now, we state the following Lemma that establish how we extend the Scherk surface in general.

Lemma 3.1. *Let u be a Scherk graph on a polygonal domain $D_1 = P(A_1, B_1, \ldots, A_k, B_k)$, where the $A_i's$ and $B_i's$ are the (geodesic) sides of ∂D_1 on which u takes values $+\infty$ and $-\infty$ respectively. Let K be a compact set in the interior of D_1. Let $D_2 = P(E_1, E_1', A_2, B_2, \ldots, A_k, B_k)$ be the polygonal domain D_1 to which we attach two regular trapezoids E_1 to the side A_1 and E_1' to the side B_1. Let $E_1(\tau)$ and $E_1'(\tau)$ be the perturbed polygons as above. Then*

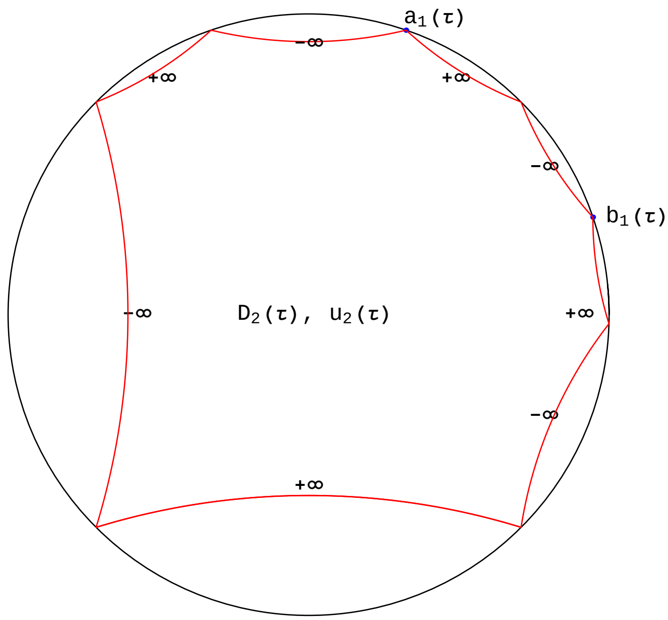


Figure 9: Perturbed Scherk domain

for all $\epsilon > 0$ *there exists* $\bar{\tau} > 0$ *so that, for all* $0 < \tau \leq \bar{\tau}$, v *is a Scherk graph on* $P(E_1(\tau), E_1'(\tau), A_2, B_2, \ldots, A_k, B_k)$ *such that*

$$\|u - v\|_{C^2(K)} \leq \epsilon. \tag{3.1}$$

Proof. The proof of this Lemma relies on [3, Section IV] with the obvious differences that we need to use the results for Scherk graphs over a domain in a Hadamard surface stated in [9] and [5]. □

Before we return to the construction, let us explain how we construct a *compact domain associated to any Scherk domain*: Let $D = P(A_1, B_1, \ldots, A_k, B_k)$ be a Scherk domain in $\mathcal{D}$ with vertex $\{v_1, \ldots, v_{2k}\} \in \partial\mathcal{D}$. Let $\beta_{v_i} : [0,1] \longrightarrow \overline{\mathcal{D}}$ denote the radial geodesic starting at $p_0 \in \mathcal{D}$ (the center of the disc $\mathcal{D}$) and ending at $v_i \in \partial\mathcal{D}$. Note that any β_{v_i} can not touch neither a A_i side nor a B_i side expect at the vertex.

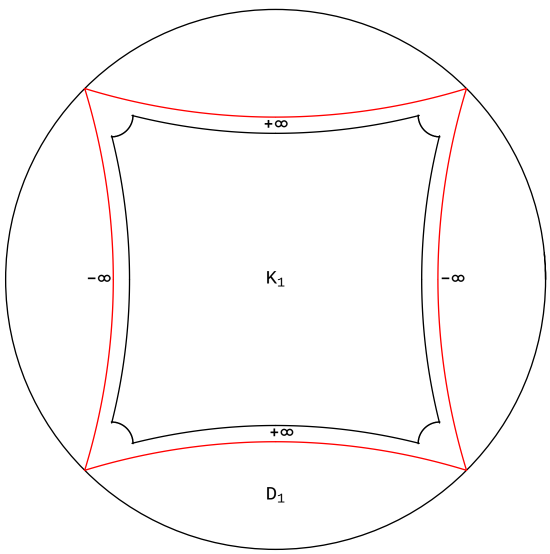
Set $r < 1$ and $p_i = \beta_{v_i}(r) \in \mathcal{D}$ for $i = 1, \ldots, 2k$. Consider the polygon

$$P = \bigcup_{i=1}^{2k-1} \gamma(p_i, p_{i+i}) \cup \gamma(p_{2k}, p_1) \subset D,$$

and let K' be the closure of the domain bounded by P, here $\gamma(p_i, p_{i+1})$ is the geodesic arc joining p_i and p_{i+1} in D. Let $\mathcal{D}(p_i, 1-r)$ be geodesic disc centered at p_i of radius $1-r$ for each $i = 1, \ldots, 2k$. Then,

Definition 3.3. *For $r < 1$ close to 1, the* **compact domain associated to the Scherk domain** *D is given by*

$$K = K' \setminus \bigcup_{i=1}^{2k} \mathcal{D}(p_i, 1-r).$$



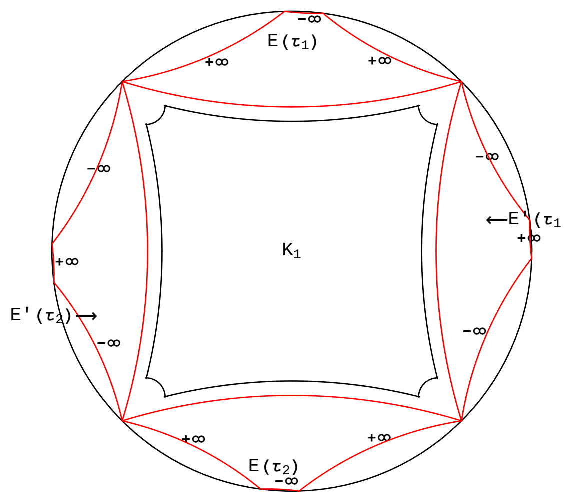


Figure 10: (Left) Compact domain associated to the inscribed quadrilateral

Figure 11: (Right) Attaching perturbed regular trapezoids

Now, we continue with the construction. Let $D_1 = P(A_1, B_1, A_2, B_2)$ be the inscribed square in $\mathcal{D}$ (given in Definition 3.1), and the Scherk graph u_1 on D_1 which is $+\infty$ on the $A_i's$ sides and $-\infty$ on the $B_i's$ sides. Let K_1 be the compact domain associated to D_1 (see Definition 3.3). We choose $r_1 < 1$ close enough to one so that $u_1 > 1$ on the geodesic sides of ∂K_1 closer to the $A_i's$ sides and $u_1 < -1$ on the geodesic sides of ∂K_1 closer to the $B_i's$ sides (cf. Figure 10).

Next, we attach perturbed regular trapezoids to the sides A_1 and B_1, so from Lemma 3.1, for any $\epsilon_2 > 0$ there exists $\tau_2 > 0$ so that $D_2(\tau) = D_1 \cup E_1(\tau) \cup E_1'(\tau)$ is a Scherk domain and $u_2(\tau)$, the Scherk graph defined on $D_2(\tau)$, satisfy

$$\|u_1 - u_2(\tau)\|_{C^2(K_1)} \le \epsilon_2,$$

for all $0 < \tau \leq \tau_2$. Moreover, we can choose $u_2(\tau)$ so that $u_1(p_0) = u_2(\tau)(p_0)$ (here p_0 is the center of $\mathcal{D}$). Then, choose $\epsilon_2 > 0$ so that $u_2(\tau) > 1$ on the geodesic sides of ∂K_1 closer to the $A_i's$ sides and $u_2(\tau) < -1$ on the geodesic sides of ∂K_1 closer to the $B_i's$ sides.

Let $K_2(\tau)$ be the compact domain associated to the Scherk domain $D_2(\tau)$. Choose $r_2 < 1$ close enough to one (in the definition of $K_2(\tau)$ given by Definition 3.3) so that, for $0 < \tau \leq \tau_2$, $u_2(\tau) > 2$ on those geodesic sides of $\partial K_2(\tau)$ parallel to the sides of $D_2(\tau)$ where $u_2(\tau) = +\infty$, and $u_2(\tau) < -2$ on the sides of $\partial K_2(\tau)$ parallel to sides of $D_2(\tau)$ where $u_2(\tau) = -\infty$ (cf. Figure 12).

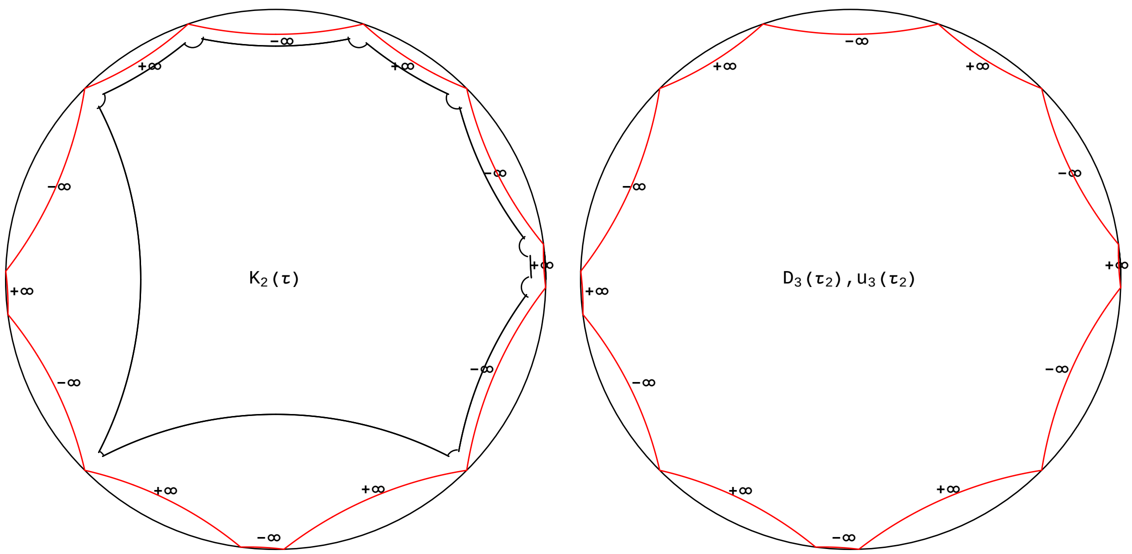


Figure 12: (Left) Compact domain associated to $D_2(\tau)$

Figure 13: (Right) Choosing $u_3(\tau)$

Continue by constructing the Scherk domain $D_3(\tau)$ by attaching perturbed regular trapezoids (as above) to the sides A_2 and B_2 of D_1. We know, for $\epsilon_3 > 0$, that there exist $\tau_3 > 0$ so that if $0 < \tau \leq \tau_3$ then the Scherk graph $u_3(\tau)$ exists, $u_3(\tau)(p_0) = u_1(p_0)$ and

$$\|u_3(\tau) - u_2(\tau)\|_{C^2(K_2(\tau))} \leq \epsilon_3.$$

Moreover, choose $\epsilon_3 > 0$ so that $u_3(\tau) > 3$ on the geodesic sides of $\partial K_2(\tau)$ closer to the $A_i's$ sides and $u_3(\tau) < -3$ on the geodesic sides of $\partial K_2(\tau)$ closer to the $B_i's$ sides (cf. Figure 13).

Now choose $\epsilon_n \longrightarrow 0$, $\tau_n \longrightarrow 0$, $K_n(\tau_n)$ so that $K_n(\tau_n) \subset K_{n+1}(\tau_{n+1})$, $\bigcup_n K_n(\tau_n) = \mathcal{D}$. Then the $u_n(\tau_n)$ converge to a graph u on $\mathcal{D}$.

To see u has the desired properties, we refer the reader to [3, pages 13 and 14] with the only difference that we need to use now Theorem 2.1.

Remark 3.1. *The above construction can be carried out in a more general situation. Actually, if we ask that*

- *The geodesic disc $\mathcal{D}$ has strictly convex boundary.*
- *There is a unique minimizing geodesic joining any two points of the disc.*

Then, we can extend the above example.

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